Mathematics Prize 1968
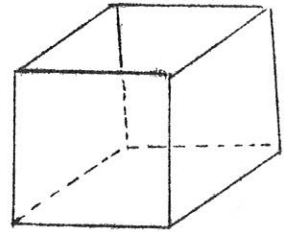
1 Consider a cubical box without a top.
What does such a box look like flattened out?
How many essentially different ways are there of drawing the
net of this box? Sketch all of them.

2. (i) Express 2.7121212...... as a rational number in its simplest form.
  (ii) Prove that there is not a next larger or a next smaller rational
       number than a given rational number.

3. The sum of consecutive positive integers beginning with 1 is called a
   triangular number (e.g. $6 = 1 + 2 + 3$ is a triangular number). Prove
   that if T is a triangular number then $8T + 1$ is a perfect
   square.
   Copy the 5 by 5 grid on to your paper and subdivide
   the grid in such a way as to illustrate the above
   result geometrically.
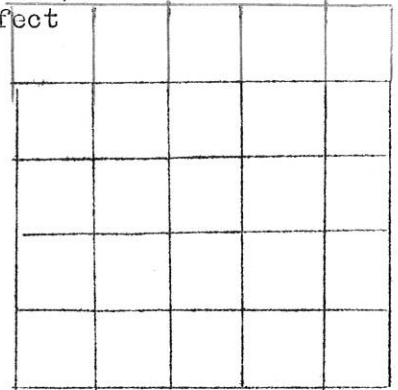
4. (i) From any point inside an equilateral triangle, prove that the sum of the
   perpendiculars to the three sides equals the altitude of the triangle.

   (ii) Two unequal circles are given intersecting at A and B. Using the transformation
   of rotation or otherwise, show how to draw a line ℓ through A such that the circles
   cut off equal chords on ℓ .

5. A man walking along a railway line has walked two-thirds of the distance across a
   bridge $1\frac{1}{2}$ miles long when he sees a train approaching at 45 m.p.h. If he can just
   manage to escape by running at uniform speed to either end of the bridge, at what
   speed must he run to avoid the train? Does it make any difference how long the bridge
   is? Give reasons.

6. You have six different colours. In how many essentially different ways could you
   paint the faces of a cube, using one colour for each face and using each colour
   once only?

7. Working in arithmetic modulo 3, one can graph "lines" on a lattice of 9 points
   e.g. the graph of $x + y = 1$ is the set of points $(0,1)$
   $(1,0)$, $(2,2)$ as shown.
   a) There are two lines "parallel" to $x + y = 1$. Give their
      equations and sketch their graphs.
   b) How many different lines are there in the 9-point plane?
      Give their equations.
   c) What is the graph of $y = x^2 + 2$?
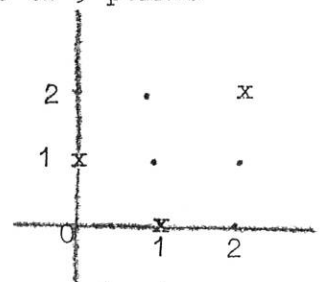
8. (i) Two circles are concentric. A tangent to the inner circle forms a chord
   12 inches long in the larger circle. Find the area of the ring between the circles.

   (ii)   A cylindrical hole is bored through a solid sphere, the central axis of
   the hole passing through the centre of the sphere. If the hole is 12 inches long,
   what is the volume of the remaining portion of the sphere?