6.3 Answers

Answer 1

$$2x - 9 \equiv 5 \pmod 8$$

$$2x \equiv 14 \pmod 8$$

$$2x \equiv 14,\ 22,\ 30,\ \ldots \pmod 8$$

$$x \equiv 7,\ 3 \pmod 8$$

This could be more succinctly written as $x \equiv 3 \pmod 4$

[3 marks]

Answer 2

$\circ$	a	b	c	d
a	c	d	c	b
b	d	c	b	a
c	a	d	c	d
d	b	a	b	c

(a) A necessary condition for the set to form a group in that its elements form a Latin Square in the binary operation table. In other words each element occurs once and once only in each row and in each column. This is not so in this case. For example, c occurs twice in the top row.

[1 mark]

(b) **(i)** $c \circ d = d$

[1 mark]

(ii) $a \circ (c \circ b) = a \circ d = b$

[1 mark]

(iii) $(a \circ c) \circ b = c \circ b = b$

[1 mark]

(iv) $(d \circ c) \circ (b \circ a) = b \circ d = a$

[1 mark]

(c) **(i)** No

[1 mark]

(ii) How the expression is bracketed makes a difference to the answer. In other words, associativity does not hold.
On the one hand, $a \circ (b \circ c) = a \circ b = d$
but on the other, $(a \circ b) \circ c = d \circ c = b$
To make sense brackets need to be present to show which of the two possible interpretations is wanted.

[1 mark]

Answer 3

(i)

$\times_{14}$	1	3	5	9	11	13
1	1	3	5	9	11	13
3	3	9	1	13	5	11
5	5	1	11	3	13	9
9	9	13	3	11	1	5
11	11	5	13	1	9	3
13	13	11	9	5	3	1

[3 marks]

(ii) The given group, $(G, \times_{14})$, has order 6.

By Lagrange's theorem, the order of a subgroup of must divide the order of the group of which it is a subgroup.

$\therefore$ $(G, \times_{14})$ can only have subgroups of order 1, 2, 3 or 6

$\therefore$ $(G, \times_{14})$ has no subgroup of order 4 ☐

[2 marks]

(iii) $H = \{ 1, 13 \}$

$\times_{14}$	1	13
1	1	13
13	13	1

[1 mark]

(iv)

$\times_{14}$	1	9	11
1	1	9	11
9	9	11	1
11	11	1	9

[3 marks]

Answer 4

(a) The identity element is the “do nothing” permutation;

$$e = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}$$

[1 mark]

(b) To undo the given permutation;

$$a^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 1 & 2 \end{pmatrix}$$

[1 mark]

(c) We need to demonstrate that $a \circ \{ b \circ c \} = \{ a \circ b \} \circ c$

$$\begin{aligned} \text{RHS} &= a \circ \{ b \circ c \} \\ &= \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 2 & 1 \end{pmatrix} \circ \left\{ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix} \right\} \\ &= \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 2 & 1 \end{pmatrix} \circ \left\{ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 3 \end{pmatrix} \right\} \\ &= \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \text{LHS} &= \left\{ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 2 & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \end{pmatrix} \right\} \circ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix} \\ &= \left\{ \begin{Bmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{Bmatrix} \right\} \circ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix} \\ &= \text{RHS} \qquad \square \end{aligned}$$

[2 marks]

(d) S_4 is of order 24. (From 4! = 24)
Possible subgroups, by Lagrange's Theorem, will be factors of 24.
As 4 is a factor of 24, it is possible that there is a subgroup of order 4.

[2 marks]

Answer 5

(a) Looking for the power of **B** that gives the identity matrix **I**;

$$\mathbf{B} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \neq \mathbf{I}$$

$$\mathbf{B}^2 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \neq \mathbf{I} \qquad \text{(mod 2)}$$

$$\mathbf{B}^3 = \mathbf{B}^2\mathbf{B} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I} \qquad \text{(mod 2)}$$

Thus, $|\mathbf{B}| = 3$, as expected.

[2 marks]

(b) $|\mathbf{I}| = 1$, $|\mathbf{A}| = 2$, $|\mathbf{C}| = 2$, $|\mathbf{D}| = 2$, $|\mathbf{E}| = 3$

[3 marks]

(c) **(i)** To be a cyclic group there must be at least one element that generates the whole group.
No element generated the whole group.
In other words, no element was of order 6 (From part (b))
So *G* is not isomorphic to a cyclic group of order 6.

(ii) A regular hexagon has 12 symmetries.
(6 rotational and 6 lines of mirror symmetry)
So the group of these symmetries will be of order 12.
Our group, *G*, is of order 6.
A necessary condition for two groups to be isomorphic is that they are of the same order. So *G* is not isomorphic to the group of symmetries of a regular hexagon.

[2 marks]

(d) The six possible matchings of elements are given in the following table (only one is required)

I	**A**	**B**	**C**	**D**	**E**
e	*c*	*a*	*d*	*f*	*b*
e	*d*	*a*	*f*	*c*	*b*
e	*f*	*a*	*c*	*d*	*b*
e	*c*	*b*	*f*	*d*	*a*
e	*d*	*b*	*c*	*f*	*a*
e	*f*	*b*	*d*	*c*	*a*

[3 marks]

Answer 6

(a) $\mathbb{Z}_3 \times \mathbb{Z}_2 = \{(0,0), (0,1), (1,0), (1,1), (2,0), (2,1)\}$

[3 marks]

(b) $\langle(1,1)\rangle = \{(1,1), (2,0), (0,1), (1,0), (2,1), (0,0)\}$
(Listed in the order they are generated)

[2 marks]

(c) As $\langle(1,1)\rangle$ has generated the whole group, $\mathbb{Z}_3 \times \mathbb{Z}_2 \cong \mathbb{Z}_6$ the cyclic group of order 6.

[2 marks]