

6.1 Trigonometric Identities, Type 2

Previously, identities were derived where a trigonometric function was applied to a multiple of the angle θ and it was equated to an expression containing powers of trigonometric functions in θ .

For example,

$$\cos 7\theta = 64 \cos^7 \theta - 112 \cos^5 \theta + 56 \cos^3 \theta - 7 \cos \theta \quad (\text{Type 1})$$

This can be done the other way around, by which is meant a power of a trigonometric function in θ is equated to a power free expression of trigonometric functions of multiples of the angle θ .

For example,

$$\cos^4 \theta = \frac{1}{8} (\cos 4\theta + 4 \cos 2\theta + 3) \quad (\text{Type 2})$$

6.2 A Pair of Essentials

In finding such Type 2 identities the following pair of results is invaluable,

The Type 2 Key

$$\bullet \quad z^n + \frac{1}{z^n} = 2 \cos(n\theta) \qquad \bullet \quad z^n - \frac{1}{z^n} = 2i \sin(n\theta)$$

Proof

$$\begin{aligned} \text{LHS} &= z^n + \frac{1}{z^n} \\ &= z^n + z^{-n} \\ &= (\cos \theta + i \sin \theta)^n + (\cos \theta + i \sin \theta)^{-n} \\ &= \cos(n\theta) + i \sin(n\theta) + \cos(-n\theta) + i \sin(-n\theta) && \text{by de Moivre's theorem} \\ &= \cos(n\theta) + i \sin(n\theta) + \cos(n\theta) - i \sin(n\theta) && \text{even and odd functions} \\ &= 2 \cos(n\theta) \\ &= \text{RHS} \qquad \square \end{aligned}$$

A proof of the other result is very similar and is left as an exercise for the reader

6.3 Example

Use the appropriate Type 2 Key result to prove that,

$$\cos^4 \theta = \frac{1}{8} (\cos 4\theta + 4 \cos 2\theta + 3)$$

Teaching Video : <http://www.NumberWonder.co.uk/v9099/6.mp4>



[4 marks]

6.4 Exercise

*Any solution based entirely on graphical
or numerical methods is not acceptable*

Marks Available : 50

Question 1

The binomial theorem gives that $\left(z + \frac{1}{z}\right)^2 = \left(z^2 + \frac{1}{z^2}\right) + 2$

- (i) Make use of the “The Type 2 Key” to obtain a useful formula for $\cos^2 \theta$

[2 marks]

- (ii) Hence determine the exact value of $\int_0^{\frac{\pi}{2}} \cos^2 \theta \, d\theta$

[2 marks]

- (iii) In a similar manner, consider $\left(z - \frac{1}{z}\right)^2$ and use “The Type 2 Key”
to determine the exact value of $\int_0^{\frac{\pi}{6}} \sin^2 \theta \, d\theta$

[3 marks]

Question 2

- (i) Beginning with an expression for $\left(z - \frac{1}{z}\right)^3$ find the constants p and q in the identity $\sin^3 \theta = p \sin 3\theta + q \sin \theta$

[4 marks]

- (ii) Prove $\sin\left(\frac{\pi}{2} - \theta\right) = \cos \theta$ and $\sin\left(\frac{3\pi}{2} - 3\theta\right) = -\cos 3\theta$

[2 marks]

- (iii) Combine your part (i) and (ii) results to find the constants r and s in the identity $\cos^3 \theta = r \cos 3\theta + s \cos \theta$

[2 marks]

Question 3

(i) Express $\left(z^2 + \frac{1}{z^2}\right)^3$ in terms of $\cos 6\theta$ and $\cos 2\theta$

[3 marks]

(ii) Hence find constants a and b such that $\cos^3 2\theta = a \cos 6\theta + b \cos 2\theta$

[3 marks]

(iii) Hence show that $\int_0^{\frac{\pi}{6}} \cos^3 2\theta \, d\theta = k\sqrt{3}$ where k is a rational constant

[4 marks]

Question 4

Further A-Level Examination Question from June 2016, FP2, Q5 (Edexcel)

(a) Use de Moivre's theorem to show that

$$\sin^5 \theta \equiv a \sin 5\theta + b \sin 3\theta + c \sin \theta$$

where a , b and c are constants to be found

[5 marks]

(b) Hence show that $\int_0^{\frac{\pi}{3}} \sin^5 \theta \, d\theta = \frac{53}{480}$

[5 marks]

Question 5

Further A-Level Examination Question from June 2007, FP3, Q4(b)(c) (Edexcel)

- (i) Express $32 \cos^6 \theta$ in the form $p \cos 6\theta + q \cos 4\theta + r \cos 2\theta + s$
where p, q, r and s are integers

[5 marks]

- (ii) Hence find the exact value of $\int_0^{\frac{\pi}{3}} \cos^6 \theta \, d\theta$

[4 marks]

Question 6

Further A-Level Examination Question from June 2006, FP2, Q2(a)(ii) (MEI)

By considering $\left(z - \frac{1}{z}\right)^4 \left(z + \frac{1}{z}\right)^2$ find A , B , C and D such that

$$\sin^4 \theta \cos^2 \theta = A \cos 6\theta + B \cos 4\theta + C \cos 2\theta + D$$

[6 marks]

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Teachers may obtain detailed worked solutions to the exercises by email from mhh@shrewsbury.org.uk