5.1 Trigonometric Identities, Type 1

Thanks to Euler's Relation and de Moivre's theorem, the world of complex numbers is a natural setting in which to generate trigonometric identities.

One fruitful approach is to consider an expression of the form $(\cos\theta + i\sin\theta)^n$ for some integer value of *n*. By then expanding the brackets in two different ways, and setting the resulting expressions equal to each other, many useful trigonometric identities are obtained.

5.2 Example

By expanding the brackets of $(\cos\theta + \mathrm{i}\sin\theta)^2$ in two different ways, derive two well known trigonometric identities.

Teaching Video : http://NumberWonder.co.uk/v9099/5.mp4

[6 marks]

5.3 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 50

Question 1

By expanding the brackets of $(\cos\theta + i \sin\theta)^3$ in two different ways, derive the following two well known trigonometric identities, frequently obtained "the long way" in the single A-Level examination papers,

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta \quad \text{and} \quad \sin 3\theta = 3\sin\theta - 4\sin^3\theta$$

[6 marks]

Question 2

By means of the substitution $x = \cos\theta$, and making use of the trigonometric identity $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$, to find all three solutions to the equation,

$$8x^3 - 6x - 1 = 0$$

Give your answers to 3 decimal places

[5 marks]

Question 3

This question is about finding all the solution to the equation

$$\cos 5\theta + 5\cos 3\theta = 0 \qquad 0 \leqslant \theta < \pi$$

You are given that $\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$

and also that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

Give your answers to 3 decimal places

[6 marks]

Question 4

Further A-Level Examination Question from June 2014, FP2(R), Q7 (Edexcel)

(i) Use de Moivre's theorem to show that,

$$\sin 5\theta = 16 \sin^5 \theta - 20 \sin^3 \theta + 5 \sin \theta$$

[5 marks]

(b) Hence find the five distinct solutions of the equation

$$16x^5 - 20x^3 + 5x + \frac{1}{2} = 0$$

giving your answers to 3 decimal places where necessary.

[5 marks]

(c) Use the identity given in (a) to find

$$\int_0^{\frac{\pi}{4}} \left(4\sin^5\theta - 5\sin^3\theta \right) d\theta$$

expressing your answer in the form $a\sqrt{2} + b$ where a and b are rational numbers.

[4 marks]

Question 5

(i) Use de Moivre's theorem to show that

$$\sin 4\theta = 4\cos^3\theta \sin\theta - 4\cos\theta \sin^3\theta$$

[4 marks]

(ii) Hence, or otherwise, show that,

$$\tan 4\theta = \frac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}$$

[4 marks]

(iii) Use your answer to part (b) to find, to 2 decimal places, the four solutions of the equation,

$$x^4 + 4x^3 - 6x^2 - 4x + 1 = 0$$

[5 marks]

Question 6

Further A-Level Examination Question from June 2018, FP2, Q7(a) (Edexcel)

Use de Moivre's theorem to show that

$$\cos 7\theta = 64\cos^7\theta - 112\cos^5\theta + 56\cos^3\theta - 7\cos\theta$$

[6 marks]