4.1 The Merge of Trigonometry and Complex Numbers

Euler's Relation is important because it firmly links together the subjects of trigonometry and complex numbers in a fundamental way. This merging of two topics, each important in their own right, is further strengthened by a formula found by Abraham de Moivre in 1707,

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De Moivre's Theorem

For any positive integer, *n*,

$$(r(\cos\theta + \mathrm{i}\sin\theta))^n = r^n(\cos(n\theta) + \mathrm{i}\sin(n\theta))$$

The formula will also work with negative integers although not, as it stands, with fractional values of *n*. Here it will be proven for positive integers.

Proof (by induction)

Basis step :

For $n = 1$ LHS $= (r(\cos\theta + \mathrm{i}\sin\theta))^1 = r(\cos\theta + \mathrm{i}\sin\theta)$

RHS $= r^1(\cos(1\theta) + \mathrm{i}\sin(1\theta)) = r(\cos\theta + \mathrm{i}\sin\theta)$

As LHS = RHS, de Moivre's theorem is true for $n = 1$

Assumption step :

Assume that de Moivre's theorem is true for $n = k$, $k \in \mathbb{Z}^+$ in which case,

$$(r(\cos\theta + \mathrm{i}\sin\theta))^k = r^k(\cos(k\theta) + \mathrm{i}\sin(k\theta))$$

Inductive step :

When $n = k + 1$, the LHS of the theorem becomes,

$$\begin{aligned}
&(r(\cos\theta + \mathrm{i}\sin\theta))^{k+1} \\
&= (r(\cos\theta + \mathrm{i}\sin\theta))^k(r(\cos\theta + \mathrm{i}\sin\theta))^1 \\
&= r^k(\cos(k\theta) + \mathrm{i}\sin(k\theta))\, r(\cos\theta + \mathrm{i}\sin\theta) \\
&= r^{k+1}(\cos(k\theta)\cos\theta - \sin(k\theta)\sin\theta + \mathrm{i}(\sin(k\theta)\cos\theta + \cos(k\theta)\sin\theta)) \\
&= r^{k+1}(\cos(k\theta + \theta) + \mathrm{i}\sin(k\theta + \theta)) \\
&= r^{k+1}(\cos((k+1)\theta) + \mathrm{i}\sin((k+1)\theta))
\end{aligned}$$

which is the RHS of the theorem with $n = k + 1$

Conclusion step:

Thus, if the theorem is true for $n = k$, then it is true for $n = k + 1$

As the theorem is true for $n = 1$, it is true for all positive integers, by induction □

Teaching Video : http://www.NumberWonder.co.uk/v9099/4.mp4

The video talks through the proof of de Moivre's theorem on the previous page

Abraham de Moivre
(1667-1754)
A French mathematician known for de Moivre's theorem, a formula that links complex numbers and trigonometry, and for his statistics work with the normal distribution.
He wrote a book on probability theory, “The Doctrine of Chances”, much prized by gamblers.
He moved to England to escape religious persecution, there becoming a friend of Isaac Newton, Edmond Halley and James Sterling.
He was the first to postulate the Central Limit Theorem, a cornerstone of probability theory.

4.2 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 40

Question 1

$$z = \sqrt{2}\left(cos\left(\frac{\pi}{6}\right) + \mathrm{i}\, sin\left(\frac{\pi}{6}\right)\right)$$

Find the exact value of z^5, giving your answer in the form $a + \mathrm{i}\, b$ where $a,\ b \in \mathbb{R}$

[3 marks]

Question 2

$$w = 2\left(cos\left(\frac{\pi}{3}\right) + \mathrm{i}\, sin\left(\frac{\pi}{3}\right)\right)$$

Find the exact value of w^4, giving your answer in the form $a + \mathrm{i}\, b$ where $a,\ b \in \mathbb{R}$

[3 marks]

Question 3

$$v = \sqrt{3}\left(cos\left(\frac{3\pi}{4}\right) - \mathrm{i}\, sin\left(\frac{3\pi}{4}\right)\right)$$

Find the exact value of v^6, giving your answer in the form $a + \mathrm{i}\, b$ where $a,\ b \in \mathbb{R}$

[3 marks]

Question 4

$$z = -2 + (2\sqrt{3})\mathrm{i}$$

(i) Express z in the polar form $r(\cos\theta + \mathrm{i}\sin\theta)$ where r and θ have values that you have determined.

[3 marks]

Using de Moivre's theorem,

(ii) show that z^6 is a real number, and state its numerical value

[2 marks]

Question 5

Express $(3 + \sqrt{3}\,\mathrm{i})^5$ in the form $a + b\sqrt{3}\,\mathrm{i}$ where a and b are integers

[4 marks]

Question 6
De Moivre's theorem can be used to establish to following useful result,

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Exponential Form, Powered

$$\left(r\, e^{i\theta}\right)^{n} = r^{n}\, e^{in\theta}$$

Prove this result without using any laws of indices, but using de Moivre's theorem

[4 marks]

Question 7
Using de Moivre's theorem or any of the laws of indices previously proven,

prove that $\dfrac{\left(\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right)^{6}}{\left(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right)^{11}} = \cos\left(\dfrac{\pi}{4}\right) - i\sin\left(\dfrac{\pi}{4}\right)$

[6 marks]

Question 8

(i) Express $\dfrac{1 + \sqrt{3}\,\mathrm{i}}{1 - \sqrt{3}\,\mathrm{i}}$ in the form $r\,e^{\mathrm{i}\theta}$, where $r > 0$ and $-\pi < \theta \leqslant \pi$

[4 marks]

(ii) Hence find the smallest positive integer value of n for which

$$\left(\frac{1 + \sqrt{3}\,\mathrm{i}}{1 - \sqrt{3}\,\mathrm{i}}\right)^{n}$$

is real and positive

[3 marks]

Question 9

Determine, in as simple a form as possible, the value of

$$\frac{18(1 - i)^{39}}{(1 + i)^{41}}$$

[5 marks]