2.1 Tales Of The Infinite

For real x, the exponential function, e^x, can be written as the series expansion,

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \dots + \frac{x^r}{r!} + \dots \qquad x \in \mathbb{R}$$

The utility of this function is enhanced by defining it to also apply when x is a complex number and exploring the consequences.

Of particular interest is how the series expansion can be manipulated when x is the complex number, $x = \mathrm{i}\theta$ where θ is a real number constant.

$$\begin{aligned} e^{\theta \mathrm{i}} &= 1 + \mathrm{i}\,\theta + \frac{(\mathrm{i}\theta)^2}{2!} + \frac{(\mathrm{i}\theta)^3}{3!} + \frac{(\mathrm{i}\theta)^4}{4!} + \frac{(\mathrm{i}\theta)^5}{5!} + \frac{(\mathrm{i}\theta)^6}{6!} + \dots \\ &= 1 + \theta\mathrm{i} + \frac{\mathrm{i}^2\theta^2}{2!} + \frac{\mathrm{i}^3\theta^3}{3!} + \frac{\mathrm{i}^4\,\theta^4}{4!} + \frac{\mathrm{i}^5\,\theta^5}{5!} + \frac{\mathrm{i}^6\,\theta^6}{6!} + \dots \\ &= 1 + \theta\mathrm{i} - \frac{\theta^2}{2!} - \frac{\mathrm{i}\,\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{\mathrm{i}\,\theta^5}{5!} - \frac{\theta^6}{6!} + \dots \\ &= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \dots\right) + \mathrm{i}\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \dots\right) \\ &= \cos\theta + \mathrm{i}\sin\theta \end{aligned}$$

Euler's Relation

$$e^{\mathrm{i}\theta} = \cos\theta + \mathrm{i}\sin\theta$$

2.2 Exponential Form

The everyday use of Euler's Relation is in writing complex numbers that are in

- rectangular form $z = a + ib$ Also called Cartesian form

or
- polar form $z = r(\cos\theta + \mathrm{i}\sin\theta)$ Also called Trigonometric form

into the exponential form,

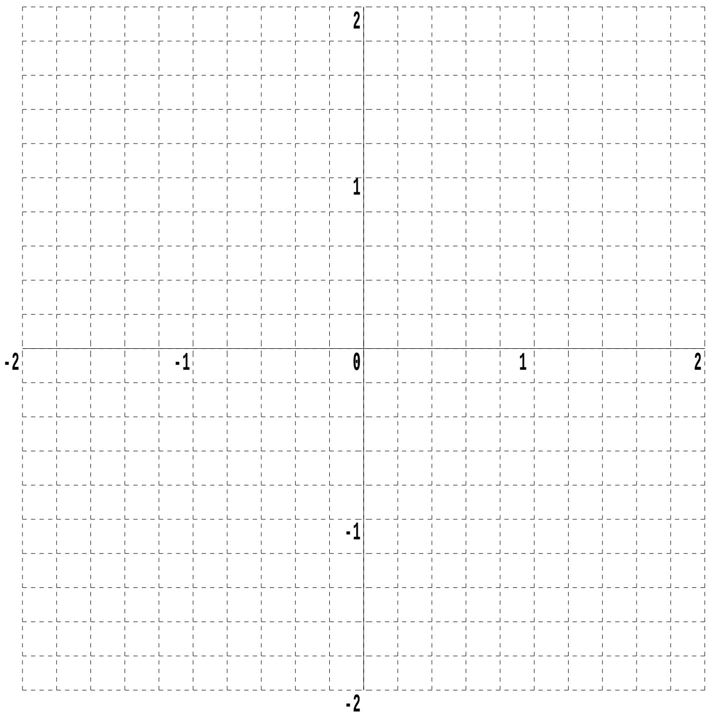
$$z = r\,e^{\mathrm{i}\theta} \text{ where } -\pi < \theta \leqslant \pi$$

Note : $r = |z|$ and $\theta = arg(z)$

2.3 Examples

Plot the following on an Argand diagram, then express them in exponential form,

(i) $-1 - \sqrt{3}\,\text{i}$ **(ii)** $\sqrt{3}\left(\cos\left(\frac{\pi}{6}\right) - \text{i}\,\sin\left(\frac{\pi}{6}\right)\right)$



Teaching Video : http://www.NumberWonder.co.uk/v9099/2a.mp4
http://www.NumberWonder.co.uk/v9099/2b.mp4

<= Part 1 Part 2 =>

[3, 3 marks]

2.4 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 56

Question 1

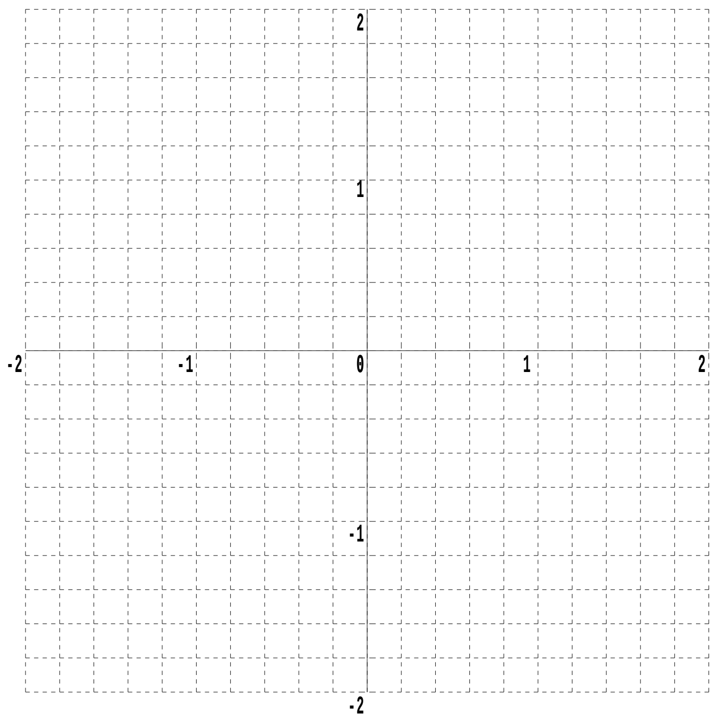
Plot the following on an Argand diagram, then express them in exponential form,

(i) $z_A = \sqrt{2}\left(cos\left(\frac{\pi}{4}\right) + \text{i}\,sin\left(\frac{\pi}{4}\right)\right)$

(ii) $z_B = \frac{3}{2}\left(cos\left(\frac{2\pi}{3}\right) + \text{i}\,sin\left(\frac{2\pi}{3}\right)\right)$

(iii) $z_C = \sqrt{3}\left(cos\left(\frac{5\pi}{6}\right) - \text{i}\,sin\left(\frac{5\pi}{6}\right)\right)$

(iv) $z_D = 2\left(cos\left(\frac{\pi}{5}\right) - \text{i}\,sin\left(\frac{\pi}{5}\right)\right)$



[3, 3, 3, 3 marks]

Question 2

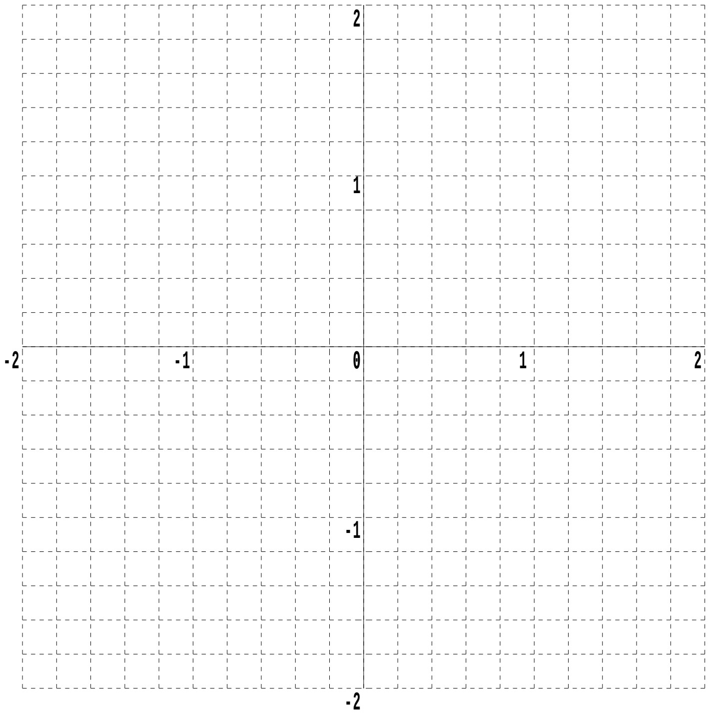
Plot the following on an Argand diagram, then express them in exponential form,

(i) $z_A = \dfrac{6}{5} + \dfrac{8}{5}\,\mathrm{i}$

(ii) $z_B = -\dfrac{3}{2} + \dfrac{1}{2}\,\mathrm{i}$

(iii) $z_C = -2\,\mathrm{i}$

(iv) $z_D = -1 - \dfrac{4}{3}\,\mathrm{i}$



[3, 3, 3, 3 marks]

Question 3

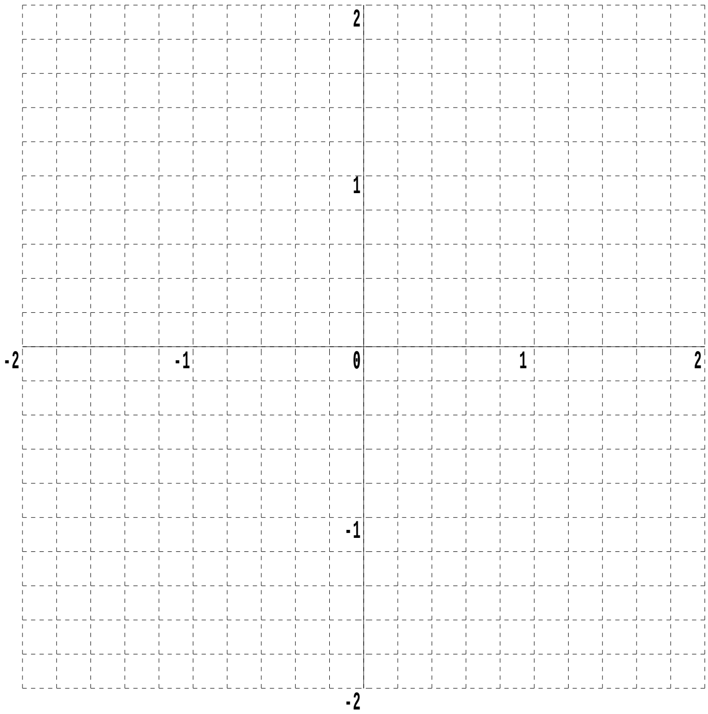
Express the following in the form $x + \mathrm{i}y$ where $x, y \in \mathbb{R}$ and also plot the points on an Argand diagram,

(i) $z_A = e^{\frac{\pi}{6}\mathrm{i}}$

(ii) $z_B = 2\,e^{\frac{3\pi}{2}\mathrm{i}}$

(iii) $z_C = \dfrac{\sqrt{2}}{2}\,e^{-\frac{\pi}{4}\mathrm{i}}$

(iv) $z_D = \dfrac{2}{3}\,e^{-\frac{4\pi}{5}\mathrm{i}}$



[3, 3, 3, 3 marks]

Question 4

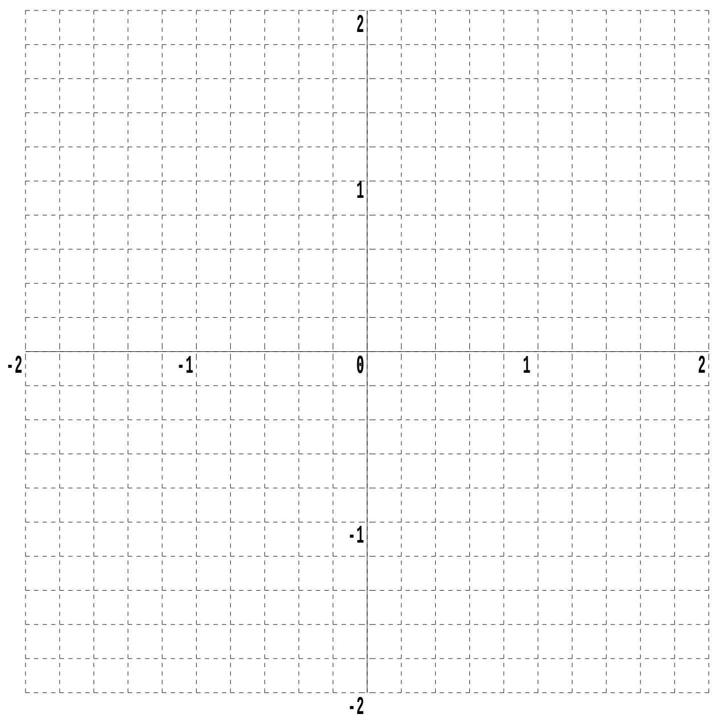
Express the following in the polar form $z = r(\cos\theta + \mathrm{i}\sin\theta)$, $-\pi < \theta \leqslant \pi$ and plot the point on an Argand diagram,

(i) $z_A = \sqrt{3}\, e^{\frac{3\pi}{2}\mathrm{i}}$

(ii) $z_B = e^{\pi \mathrm{i}}$

(iii) $z_C = \sqrt{2}\, e^{-\frac{17\pi}{4}\mathrm{i}}$

(iv) $z_D = \frac{3}{2}\, e^{-\frac{4\pi}{3}\mathrm{i}}$



[3, 3, 3, 3 marks]

Question 5

(i) Use Euler's Relation to show that $sin\,\theta \;=\; \frac{1}{2\text{i}}\left(e^{\text{i}\theta} - e^{-\text{i}\theta}\right)$

[4 marks]

(ii) Suggest a similar result for $cos\,\theta$ and demonstrate its validity, again by using Euler's Relation

[4 marks]