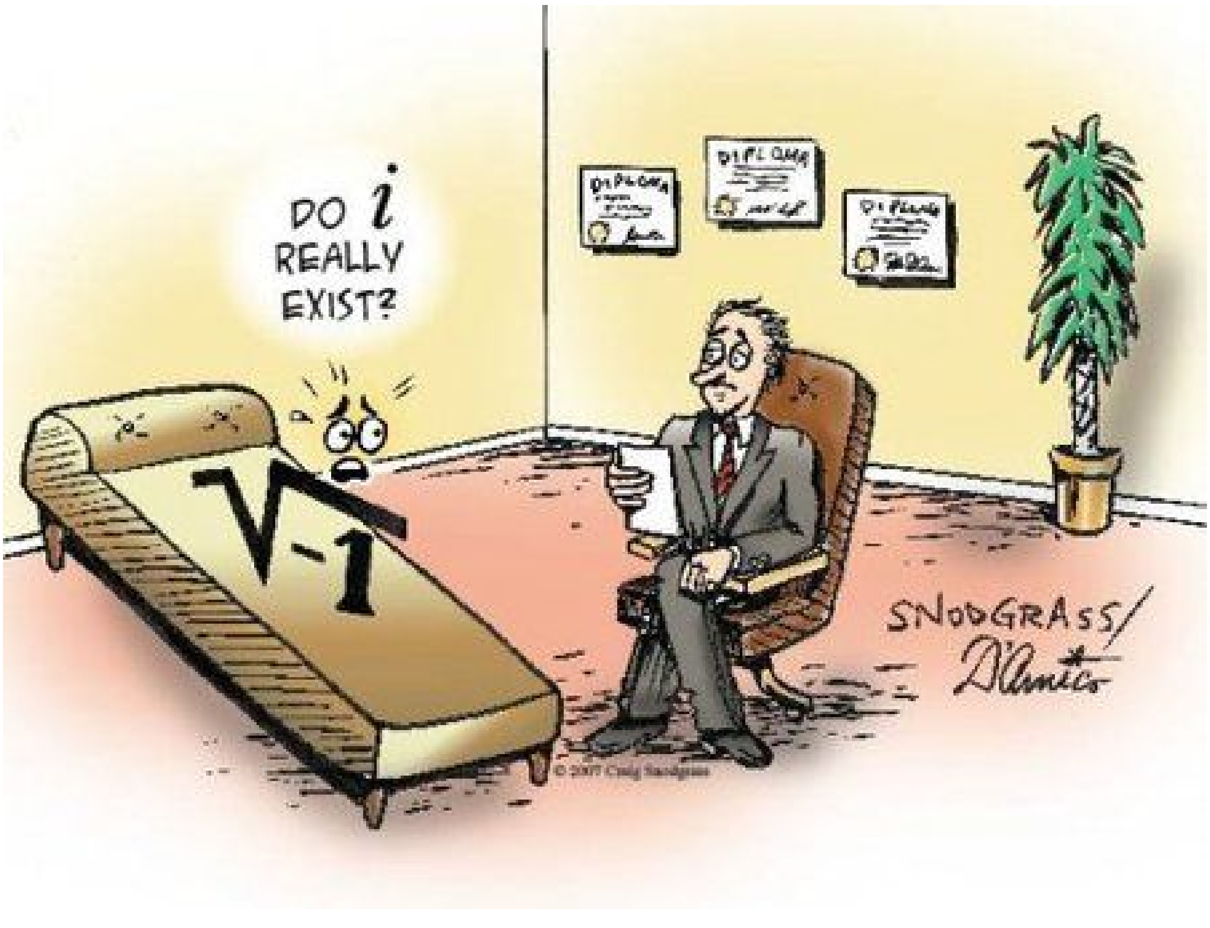
Further Pure A-Level Mathematics
Compulsory Course Component
Core 2

COMPLEX NUMBERS II



Cartoon by Craig Snodgrass

COMPLEX NUMBERS II

Lesson 1

Further A-Level Pure Mathematics, Core 2
Complex Numbers II

1.1 Odd and Even Functions

Functions can be usefully classified as being one of three types,

- Odd
- Even
- Neither odd nor even

The terminology comes from the fact that $f(x) = x^n$ is an odd function if n is odd and an even function if n is even.

Odd Functions

If a function $f(x)$ has the property that $f(-x) = -f(x)$
then that function is described as being "odd"
It's graph has half-turn rotational symmetry about the origin

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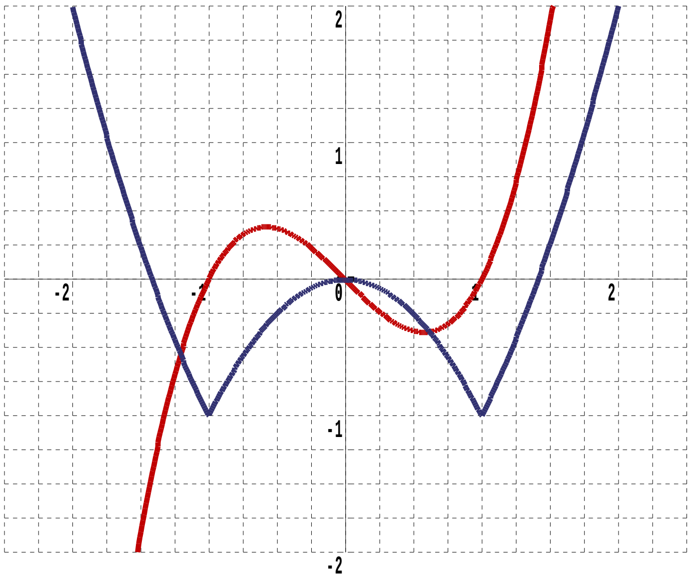
Even Functions

If a function $f(x)$ has the property that $f(-x) = f(x)$
then that function is described as being "even"
It's graph has mirror symmetry in the y-axis

1.2 Example



The red odd function $f(x) = x^3 - x$ has half-turn rotational symmetry about the origin

The blue even function $g(x) = \left|x^2 - 1\right| - 1$ has mirror symmetry in the y-axis

1.3 Odd and Even Proofs

Prove that **(i)** $f(x) = x^3 - x$ is an odd function

(ii) $g(x) = \left| x^2 - 1 \right| - 1$ is an even function

Teaching Video : http://www.NumberWonder.co.uk/v9099/1.mp4

[4 marks]

1.4 Exercise

Any solution based entirely on graphical or numerical methods is not acceptable

Marks Available : 50

Question 1

Prove that $n(x) = \dfrac{3x^4 + 1}{x}$ is an odd function

[3 marks]

Question 2

$$h(x) = \sqrt{1 + x + x^2} + \sqrt{1 - x + x^2}$$

Prove that $h(x)$ is an even function

[4 marks]

Question 3

(i) What feature of the graph of $f(x) = sin\,x$ suggests that the sine function is an odd function ?

[1 mark]

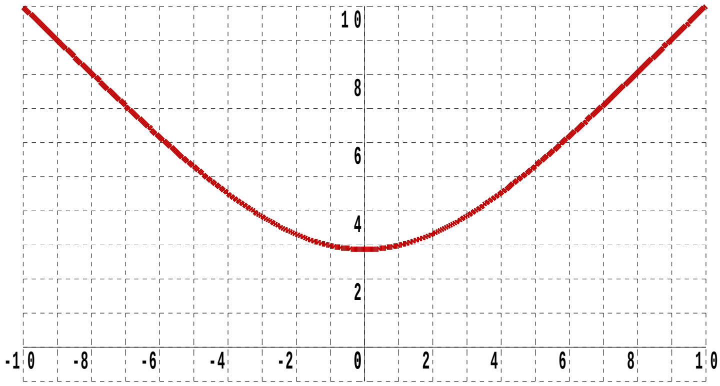
(ii) Prove that $f(x) = sin\,x$ is an odd function by making use of its Maclaurin series

[4 marks]

Question 4

$$v(x) = x \frac{2^x + 1}{2^x - 1}$$

The graph of $v(x)$ suggests that it may be an even function,



Prove that $v(x)$ is indeed an even function

[4 marks]

Question 5

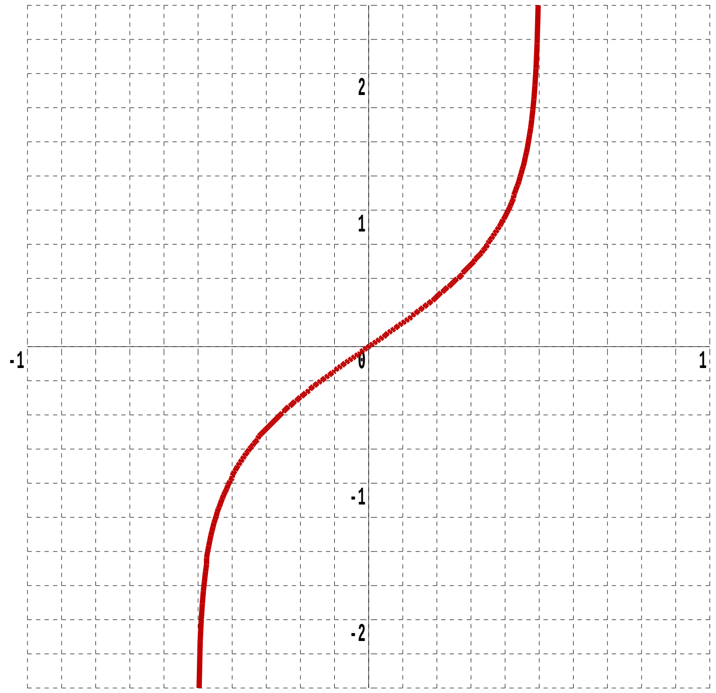
Prove that $s(x) = \frac{(1 + e^x)^2}{e^x}$ is an even function

[5 marks]

Question 6

$$L(x) = ln\left(\frac{0.5 + x}{0.5 - x}\right) \qquad x \in \mathbb{R}, \quad -0.5 < x < 0.5$$

The graph of $L(x)$ suggests that it may be an odd function,



Prove that $L(x)$ is indeed an odd function

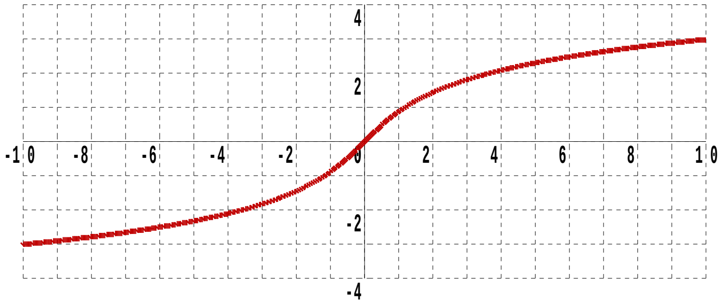
[5 marks]

Question 7

It can be shown† that,

- the derivative of an odd function is an even function
- the derivative of an even function is an odd function

The graph of $k(x) = ln\left(x + \sqrt{1 + x^2}\right)$ is given below and suggests that this may be an odd function



(i) Show that the derivative of $k(x)$ is $\dfrac{1}{\sqrt{1 + x^2}}$

[6 marks]

(ii) Hence prove that $k(x)$ is indeed an odd function

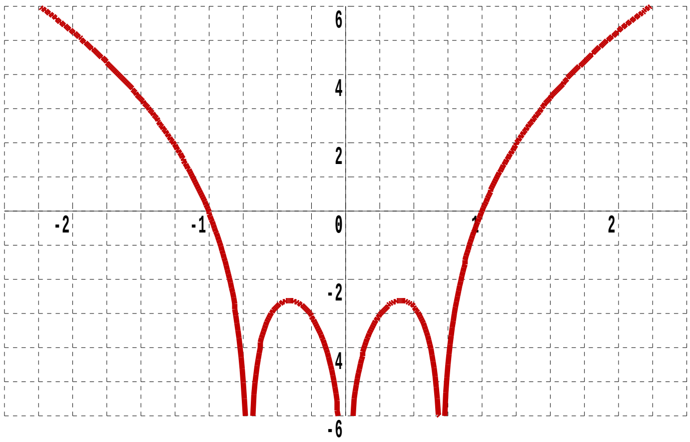
[2 marks]

† See Question 9

Question 8

$$p(x) = ln\left(2x^3 - x\right)^2 \qquad x \in \mathbb{R},\ x \neq 0,\ x \neq \pm\sqrt{\frac{1}{2}}$$

The graph of $p(x)$ suggests that it may be an even function,



Prove that $p(x)$ is indeed an even function by,

(i) First showing that $p'(x)$ is an odd function

[4 marks]

(ii) only algebra and no differentiation

[4 marks]

Question 9

(i) Study the following proof,

- The derivative of an odd function is an even function

Suppose that $f(x)$ is an odd function in which case $f(-x) = -f(x)$

$$\therefore \quad -f(x) = f(-x)$$

differentiate both sides

$$-f'(x) = f'(-x) \times (-1) \text{ by the chain rule}$$

$$-f'(x) = -f'(-x)$$

That is, $f'(x) = f'(-x)$

Which is saying the derivative of f (which was an odd function) is an even function ☐

(ii) Prove the following,

- The derivative of an even function is an odd function

[4 marks]

Question 10

Given a function $f(x)$ give a reason why $f(|x|)$ is guaranteed to be an even function but $|f(x)|$ is not.

[4 marks]