

**3.1 The Method of Differences**

Thus far, when the Method of Differences has been employed, the differences have occurred somewhat accidentally as a result of applying partial fractions to problems. The concept can be more purposefully set up and deployed to powerful effect.

First, however, a formal statement of exactly what the method of differences is:

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**The Method of Differences**

If the general term,  $u_r$ , of a series can be expressed in the form

$$f(r + 1) - f(r)$$

then 
$$\sum_{r=1}^n u_r = \sum_{r=1}^n (f(r + 1) - f(r))$$

In consequence, 
$$u_1 = f(2) - f(1)$$

$$u_2 = f(3) - f(2)$$

$$u_3 = f(4) - f(3)$$

...

$$u_n = f(n + 1) - f(n)$$

$$\therefore \sum_{r=1}^n u_r = f(n + 1) - f(1)$$


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### 3.2 Example

Use the method of differences to find a formula for,

$$\sum_{r=1}^n r(r+1)(r+2)$$

Begin by showing that,

$$r(r+1)(r+2) = \frac{1}{4} (r(r+1)(r+2)(r+3) - (r-1)r(r+1)(r+2))$$

Teaching Video : <http://www.NumberWonder.co.uk/v9097/3.mp4>



[ 6 marks ]

### 3.3 Exercise

*Any solution based entirely on graphical  
or numerical methods is not acceptable*

Marks Available : 50

#### Question 1

( a ) Prove that  $r(r + 1) - (r - 1)r = 2r$

[ 2 mark ]

( b ) Hence prove, by the method of differences, that

$$\sum_{r=1}^n r = \frac{1}{2}n(n + 1)$$

[ 4 marks ]

## Question 2

Use the facts that

- $(r + 1)^3 - r^3 = 3r^2 + 3r + 1$
- $\sum_{r=1}^n r = \frac{1}{2}n(n + 1)$  ( Proven in Question 1 )
- $\sum_{r=1}^n 1 = n$  ( Obvious ! )

to construct a proof, using the method of differences, that

$$\sum_{r=1}^n r^2 = \frac{1}{6}n(n + 1)(2n + 1)$$

[ 6 marks ]

**Question 3**

*Further A-Level Examination Question from June 2016, FP2, Q2*

(a) Show that,

$$r^2(r+1)^2 - (r-1)^2r^2 \equiv 4r^3$$

[ 3 marks ]

Given that  $\sum_{r=1}^n r = \frac{1}{2}n(n+1)$

(b) use the identity in (a) and the method of differences to show that

$$(1^3 + 2^3 + 3^3 + \dots + n^3) = (1 + 2 + 3 + \dots + n)^2$$

[ 4 marks ]

**Question 4**

This question is about finding a formula for the series,

$$\sum_{r=1}^n (r \times r!) = 1 \times 1! + 2 \times 2! + 3 \times 3! + \dots + n \times n!$$

- (a) Prove that  $((r + 1)! - 1) - (r! - 1) = r \times r!$

[ 2 marks ]

- (b) Use the part (a) answer and the method of differences to find a formula

for  $\sum_{r=1}^n (r \times r!)$

[ 4 marks ]

- (c) Evaluate  $\sum_{r=1}^6 (r \times r!)$

[ 1 mark ]

**Question 5**

This question is about finding a formula for the series,

$$\sum_{r=1}^n (r^2 + 1)r! = 2 \times 1! + 5 \times 2! + 10 \times 3! + \dots + (n^2 + 1)n!$$

(a) Prove that  $((r + 2)! - (r + 1)!) - 2((r + 1)! - r!) = (r^2 + 1)r!$

[ 4 marks ]

(b) Use the part (a) answer and the method of differences to find a formula

for  $\sum_{r=1}^n (r^2 + 1)r!$

[ 6 marks ]

**Question 6**

*Further A-Level Examination Question from June 2011, FP2, Q4*

Given that,

$$(2r + 1)^3 = Ar^3 + Br^2 + Cr + 1$$

- (a) find the values of the constants  $A$ ,  $B$  and  $C$

[ 2 marks ]

- (b) Show that,

$$(2r + 1)^3 - (2r - 1)^3 = 24r^2 + 2$$

[ 2 marks ]

- (c) Using the result in part (b) and the method of differences, show that

$$\sum_{r=1}^n r^2 = \frac{1}{6} n(n + 1)(2n + 1)$$

[ 5 marks ]