Lesson 2

Further A-Level Pure Mathematics
Telescoping Series : Core 2

2.1 A Refusal To Telescope

The "Method of Differences" is a powerful tool in finding the sum to *n* terms of many series but, as the name implies, it relies on there being a subtraction part in any partial fraction expansion so that the resulting equivalent series telescopes.

The sum, $\sum_{r=0}^{n} \frac{2r+3}{(r+1)(r+2)}$ looks similar to those considered previously.

Yet applying the partial fraction technique does not give any subtraction part.

$$
\begin{aligned}
\sum_{r=0}^{n} \frac{2r+3}{(r+1)(r+2)} &= \sum_{r=0}^{n} \frac{1}{r+1} + \sum_{r=0}^{n} \frac{1}{r+2} \\
&= \frac{1}{1} + \frac{1}{2} \\
&+ \frac{1}{2} + \frac{1}{3} \\
&+ \frac{1}{3} + \frac{1}{4} \\
&+ \frac{1}{4} + \ldots \\
&+ \ldots + \frac{1}{n+1} \\
&+ \frac{1}{n+1} + \frac{1}{n+2} \\
&= 1 + 2\left(\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{n+1} \right) + \frac{1}{n+2}
\end{aligned}
$$

The problem identified by the failure of the method of differences to generate a telescoping series is more serious than it might at first seem for the series at the heart of the problem is The Harmonic Series,

$$H_n = \sum_{r=1}^{n} \frac{1}{r} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{n}$$

This is divergent and the partial sums have no simple formula in terms of *n*.

Teaching Video : http://www.NumberWonder.co.uk/v9097/2.mp4

The Teaching video will talk through the above example

2.2 In The Exam

In the Further A-Level examination, the questions asked will avoid problems such as that presented in the Teaching Video. Unless a question states otherwise, the default technique, ***even when not explicitly stated***, is to apply the partial fractions method.

2.3 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable

Marks Available : 34

Question 1

Further A-Level Examination Question from 2018, Mock Paper, Core 1, Q4

(a) Prove that, for all positive integers n,

$$\sum_{r=1}^{n} \frac{1}{(5r-2)(5r+3)} \equiv \frac{n}{a(bn+c)}$$

where a, b and c are integers to be determined

[5 marks]

(b) Hence, showing your working, find the exact value of

$$\sum_{r=10}^{50} \frac{1}{(5r-2)(5r+3)}$$

[2 marks]

Question 2

Further A-Level Examination Question from June 2019, Core 1, Q4

Prove that, for $n \in \mathbb{Z}, n \geqslant 0$

$$\sum_{r=0}^{n} \frac{1}{(r+1)(r+2)(r+3)} = \frac{(n+a)(n+b)}{c(n+2)(n+3)}$$

where a, b and c are integers to be found

[5 marks]

Question 3

(a) By multiplying the numerator and the denominator by the denominator's conjugate, turn the following into a telescoping series and hence obtain an expression for the sum to *n* terms,

$$\sum_{r=1}^{n} \frac{1}{\sqrt{r} + \sqrt{r+1}}$$

[4 marks]

(b) Hence evaluate,

$$\frac{1}{\sqrt{4} + \sqrt{5}} + \frac{1}{\sqrt{5} + \sqrt{6}} + \ldots + \frac{1}{\sqrt{80} + \sqrt{81}}$$

[2 marks]

Question 4

(a) Show that $\dfrac{r}{(r+1)!} = \dfrac{1}{r!} - \dfrac{1}{(r+1)!}$

[2 marks]

(b) Hence find $\displaystyle\sum_{r=1}^{n} \frac{r}{(r+1)!}$

[5 marks]

Question 5

Determine the exact value of,

$$\sum_{r=1}^{30}\left(\frac{1}{ln(r+1)}-\frac{1}{ln(r+2)}\right)$$

[5 marks]

Question 6

Show that the partial fractions method fails when trying to determine the following sum via a telescoping series,

$$\sum_{r=1}^{n} \frac{1}{(3r-2)(3r+2)}$$

Interestingly, this #FAIL example does have a subtraction component. Thus it is a step on from the #FAIL example at the start of this lesson which didn't work because there were no subtractions at all. However, this example still cannot be made to work, essentially because $3p - 2 \neq 3q + 2$ for any integers p and q.

[4 marks]