Further Pure A-Level Mathematics
Compulsory Course Component
Core 2

TELESCOPING SERIES

THE METHOD OF DIFFERENCES

T E L E S C O P I N G S E R I E S

T H E M E T H O D O F D I F F E R E N C E S

Lesson 1

Further A-Level Pure Mathematics
Telescoping Series : Core 2

1.1 The Classic Telescoping Series Example

Often in mathematics, a seemingly unremarkable question can lead to an ingenious solution with an underlying technique that is then of use in solving many similar and seeming harder questions. This in turn can leads to a focus on questions that one feels should be solvable in a likewise manner but which are not.

A excellent example of such an apparently unremarkable question is the sum,

$$\sum_{r=1}^{n} \frac{1}{r(r+1)} = \frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \ldots + \frac{1}{n(n+1)}$$

One approach is to look at the initial partial sums of the series,

$$\frac{1}{2} = \frac{1}{2}$$

$$\frac{1}{2} + \frac{1}{6} = \frac{2}{3}$$

$$\frac{1}{2} + \frac{1}{6} + \frac{1}{12} = \frac{3}{4}$$

$$\frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} = \frac{4}{5}$$

$$\frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \frac{1}{30} = \frac{5}{6}$$

A reasonable guess at this stage would be that,

$$\sum_{r=1}^{n} \frac{1}{r(r+1)} = \frac{n}{n+1}$$

An this guess could be proven correct with a proof by induction

There is, however, a degree of unsatisfactory messiness about this approach for it does not give a particularly insightful reason for why the result is true.

For the “insight” an old tool is needed, that of partial fractions, along with a willingness to explore the resulting alternative equivalent series.

Teaching Video : http://www.NumberWonder.co.uk/v9097/1a.mp4
http://www.NumberWonder.co.uk/v9097/1b.mp4

<= Part 1 Part 2 =>

After watching the video, write out the details of the Telescoping Series method of showing that,

$$\sum_{r=1}^{n} \frac{1}{r(r+1)} = \frac{n}{n+1}$$

[6 marks]

1.2 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 34

Question 1

(a) Express $\dfrac{1}{(r+2)(r+3)}$ in partial fractions

[1 mark]

(b) Hence use the Telescoping Series method to find the sum of the series,

$$\sum_{r=1}^{n} \frac{1}{(r+2)(r+3)}$$

[5 marks]

The Telescoping Series are the result of applying a technique known as
The Method of Differences

Question 2

Further A-Level Examination Question from June 2010, FP2, Q1

(a) Express $\dfrac{3}{(3r-1)(3r+2)}$ in partial fractions

[2 marks]

(b) Using your answer to part (a) and the method of differences, show that

$$\sum_{r=1}^{n} \frac{3}{(3r-1)(3r+2)} = \frac{3n}{2(3n+2)}$$

[3 marks]

(c) Evaluate $\displaystyle\sum_{r=100}^{1000} \frac{3}{(3r-1)(3r+2)}$ giving your answer to 3 significant figures

[2 marks]

Question 3

(a) Express $\dfrac{1}{r(r+2)}$ in partial fractions

[1 mark]

(b) Hence find the sum of the series $\displaystyle\sum_{r=1}^{n} \frac{1}{r(r+2)}$ using the method of differences

[5 marks]

Question 4

Further A-Level Examination Question from June 2018, IAL, F2, Q5

(a) Express $\dfrac{4r + 2}{r(r+1)(r+2)}$ in partial fractions

[3 marks]

(b) Hence using the method of differences, prove that

$$\sum_{r=1}^{n} \frac{4r + 2}{r(r+1)(r+2)} = \frac{n(an+b)}{2(n+1)(n+2)}$$

where a and b are constants to be found

[5 marks]

Question 5

Further A-Level Examination Question from June 2017, FP2, Q1

(a) Show that, for $r > 0$

$$\frac{1}{r^2} - \frac{1}{(r+1)^2} \equiv \frac{2r+1}{r^2(r+1)^2}$$

[1 mark]

(b) Hence prove that, for $n \in \mathbb{N}$

$$\sum_{r=1}^{n} \frac{2r+1}{r^2(r+1)^2} = \frac{n(n+2)}{(n+1)^2}$$

[3 marks]

(c) Show that, for $n \in \mathbb{N}, n > 1$

$$\sum_{r=n}^{3n} \frac{6r + 3}{r^2 (r + 1)^2} = \frac{a n^2 + bn + c}{n^2 (3n + 1)^2}$$

where *a, b* and *c* are constants to be found

[3 marks]