Lesson 5

Further A-Level Pure Mathematics
Proof by Induction : Core 1

5.1 From the Examination Hall

Summary

Proof by induction is used to prove that a statement is true for all positive integers.
A typical proof by induction will have four main steps;

- Basis — The statement is proved true for $n = 1$
- Assumption — The statement is assumed to be true for $n = k$
- Inductive — The statement is shown, in consequence, to be true for $n = k + 1$
- Conclusion — The statement is consequently deduced true for all positive integers n

Cases Considered

- Proving divisibility by an integer
- Proving a position-to-term formula for a sequence defined by a term to term formula
- Proving a relationship about a matrix raised to the power n
- Proving a formula for the sum of a series, often involving sigma notation

As with all proofs, what examiners are looking for is the quality of the logical reasoning and an attention to the small details

5.2 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 33

Question 1

Further AS-Level Sample Assessment Materials, Core 1, Q6 (a)

Prove by induction that for all positive integers n,

$$\sum_{r=1}^{n} r^2 = \frac{1}{6} n(n+1)(2n+1)$$

[6 marks]

Question 2

Further A-Level Sample Assessment Materials, Core 1, Q2

Prove by induction that for all positive integers n

$$f(n) = 2^{3n+1} + 3\left(5^{2n+1}\right)$$

is divisible by 17

[6 marks]

Question 3

Further A-Level Examination Question from May 2016, Q8

(i) Prove by induction that, for $n \in \mathbb{Z}^+$

$$\sum_{r=1}^{n} \frac{2r+1}{r^2(r+1)^2} = 1 - \frac{1}{(n+1)^2}$$

[5 marks]

(ii) A sequence of positive rational numbers is defined by,

$$u_1 = 3$$

$$u_{n+1} = \frac{1}{3} u_n + \frac{8}{9} \qquad n \in \mathbb{Z}^+$$

Prove by induction that, for $n \in \mathbb{Z}^+$

$$u_n = 5 \times \left(\frac{1}{3} \right)^n + \frac{4}{3}$$

[5 marks]

Question 4

Further A-Level Examination Question from May 2018, IAL F1, Q8

Prove by induction that, for $n \in \mathbb{Z}^+$

$$\begin{pmatrix} a & 0 \\ 1 & b \end{pmatrix}^n = \begin{pmatrix} a^n & 0 \\ \dfrac{a^n - b^n}{a - b} & b^n \end{pmatrix}$$

where a and b are constants and $a \neq b$

[5 marks]

Question 5

Further A-Level Examination Question from January 2017, IAL, F1, Q9 (ii)

Prove by induction that, for $n \in \mathbb{Z}^+$

$$f(n) = 5^{2n} + 3n - 1$$

is divisible by 9

[6 marks]