## 2.1 Iterative Relations and Induction

A sequence is defined iteratively by the term-to-term formula,

$$u_1 = 1, \quad u_{n+1} = 3u_n + 4$$

The first few terms of this sequence are,

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| $u_n$ | 1 | 7 | 25 | 79 | 241 | 727 | 2185 | 6559 |

It is thought that a position-to-term formula is, $u_n = 3^n - 2$

It works for the eight terms listed, but to ensure it will always work a proof is needed.

### Statement

For the sequence with the iterative definition, $u_1 = 1, \quad u_{n+1} = 3u_n + 4$

for all positive integers $n$, the $n^{th}$ term is given by $u_n = 3^n - 2$

### Proof by Induction

When $n = 1, \quad u_1 = 3^1 - 2$

$= 1$ which is the correct initial term

Assume that when $n = k, \quad u_k = 3^k - 2$ in which case...

$$\begin{aligned} u_{k+1} &= 3u_k + 4 \\ &= 3\left(3^k - 2\right) + 4 \\ &= 3^{k+1} - 6 + 4 \\ &= 3^{k+1} - 2 \end{aligned}$$

Therefore, if the result is true for $n = k$, then it is true for $n = k + 1$

As the result has been shown to be true for $n = 1$, the conclusion is that it is true for all positive integers by mathematical induction □

**[ 5 marks ]**

The Teaching Video will talk you through the above proof.

Teaching Video : http://www.NumberWonder.co.uk/v9096/2.mp4

## 2.2 Exercise

*Any solution based entirely on graphical*
*or numerical methods is not acceptable*
Marks Available : 30

**Question 1**

A sequence is defined iteratively by the term-to-term formula,

$$u_1 = 1, \quad u_{n+1} = 2u_n + 1$$

The first few terms of this sequence are,

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| $u_n$ | 1 | 3 | 7 | 15 | 31 | 63 | 127 | 255 |

Prove by induction that the position-to-term formula is $u_n = 2^n - 1$

**[ 5 marks ]**

**Did you know ?**

Numbers of the form $2^n - 1$ that are prime are called Mersenne Primes

**Question 2**

A sequence is defined iteratively by the term-to-term formula,

$$u_1 = 7, \quad u_{n+1} = 2u_n - 1$$

The first few terms of this sequence are,

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| $u_n$ | 7 | 13 | 25 | 49 | 97 | 193 | 385 | 769 |

Prove by induction that the position-to-term formula is $u_n = 3 \times 2^n + 1$

**[ 5 marks ]**

**Question 3**

A sequence is defined iteratively by the hybrid formula,

$$u_1 = 6, \quad u_{n+1} = u_n + 2^n + 4 \qquad n \geqslant 1$$

The first few terms of this sequence are,

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| $u_n$ | 6 | 12 | 20 | 32 | 52 | 88 | 156 | 288 |

Prove by induction that the position-to-term formula is $u_n = 2^n + 4n$

**[ 5 marks ]**

**Question 4**

*Further A-Level Examination Question from June 2013, IAL, FP1, Q9 (a)*

A sequence of numbers is defined by

$$u_1 = 8$$

$$u_{n+1} = 4u_n - 9n \qquad n \geqslant 1$$

Prove by induction that, for $n \in \mathbb{Z}^+$

$$u_n = 4^n + 3n + 1$$

**[ 5 marks ]**

**Question 5**

*Further A-Level Examination Question from January 2018, IAL, F1, Q8 (i)*

A sequence of numbers is defined by,

$$u_1 = 3$$

$$u_{n+1} = u_n + 3n - 2 \qquad n \geqslant 1$$

Prove by induction, for all positive integers $n$

$$u_n = \frac{3}{2}n^2 - \frac{7}{2}n + 5$$

**[ 5 marks ]**

**Question 6**

*Further A-Level Examination Question from January 2013, IAL, FP1, Q8 (b)*

A sequence of positive integers is defined by,

$$u_1 = 1$$

$$u_{n+1} = u_n + n(3n + 1) \qquad n \in \mathbb{Z}^+$$

Prove by induction that,

$$u_n = n^2(n - 1) + 1 \qquad n \in \mathbb{Z}^+$$

**[ 5 marks ]**

This document is Licensed for use by staff and students at **Shrewsbury School, England**
To obtain a licence please visit www.NumberIsAll.com
© 2020 Number Wonder, Number Is All