Further Pure A-Level Mathematics
Compulsory Course Component
Core 1

PROOF BY INDUCTION

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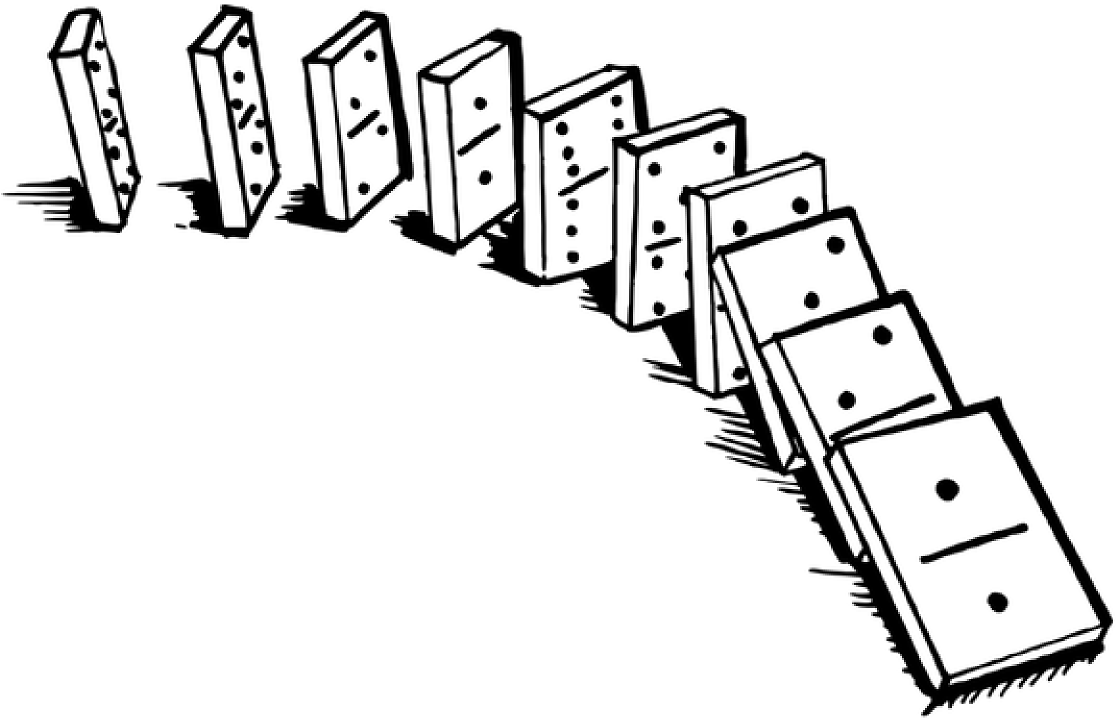
INDUCTION

Lesson 1

Further A-Level Pure Mathematics
Proof by Induction : Core 1

1.1 The Domino Effect

Proof by induction is often likened to an imaginary line of infinite extent of dominoes. To prove that all of the dominoes will fall over one must prove, firstly, that the first domino in the line will fall over and, secondly, that if any domino in the line falls over it will cause the domino after it to also fall over.



1.2 Example

Suppose that numbers in the sequence $E_n = 3^{2n} + 11$ are to be investigated.

n	1	2	3	4	5	6
E_n	20	92	740	6572	59060	531452
Prime Factorisation	$2^2.5$	$2^2.23$	$2^2.5.37$	$2^2.31.53$	$2^2.5.2953$	$2^2.132863$

From looking at the data in the table, the conjecture could be made that all numbers in this sequence are divisible by 4, and that is certainly the most striking feature for the first six terms in the sequence.
However, there is considerable uncertainly as to whether or not the property of being divisible by 4 will continue to hold for all subsequent terms.
To resolve the matter a proof is needed one way or the other.

Statement

For all positive integers n, $3^{2n} + 11$ is divisible by 4

Proof by Induction

Let $f(n) = 3^{2n} + 11, \quad n \in \mathbb{Z}^+$

When $n = 1, \quad f(1) = 3^{2 \times 1} + 11$

$= 20$ which is divisible by 4

Assume that when $n = k$, $f(k) = 3^{2k} + 11$ is divisible by 4
in which case...

$$\begin{aligned} f(k+1) &= 3^{2(k+1)} + 11 \\ &= 3^{2k} \times 3^2 + 11 \\ &= 9 \times 3^{2k} + 11 \\ &= 8\left(3^{2k}\right) + 3^{2k} + 11 \end{aligned}$$

This is clearly divisible by 4 as required, but there is an elegant extra to make that clear,

$$= 8\left(3^{2k}\right) + f(k)$$

$$f(k+1) - f(k) = 4\left(2\left(3^{2k}\right)\right)$$

$$\therefore f(k+1) \text{ is divisible by } 4$$

Therefore, if $f(n)$ is divisible by 4 when $n = k$,

then $f(n)$ is also divisible by 4 when $n = k + 1$

As $f(n)$ is divisible by 4 when $n = 1$, $f(n)$ is also divisible by 4 for all $n \in \mathbb{Z}^+$ by mathematical induction. □

[6 marks]

The Teaching Video will talk you through the above proof.

Teaching Video : http://www.NumberWonder.co.uk/v9096/1.mp4

1.3 A Useful Deduction

The Deduction Divisibility Theorem

If $f(k)$ is divisible by r and $f(k+1) - f(k)$ is divisible by r

then $f(k+1)$ is also divisible by r

1.4 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable

Marks Available : 40

Question 1

Numbers in the sequence $D_n = n^3 + 2n$ are to be investigated.

n	1	2	3	4	5	6
D_n	3	12	33	72	135	228
Prime Factorisation	3	$2^2.3$	3.11	$2^3.3^2$	$3^3.5$	$2^2.3.19$

From looking at the data it would seem that for all positive integers, n, the terms in the sequence D_n are divisible by 3.
Use mathematical induction to prove that this is indeed the case.

[6 marks]

Question 2

Numbers in the sequence $P_n = 3^{2n} - 1$ are to be investigated.

n	1	2	3	4	5	6
P_n	8	80	728	6560	59048	531440
Prime Factorisation	2^3	$2^4.5$	$2^3.7.13$	$2^5.5.41$	$2^3.11^2.61$	$2^4.5.7.13.73$

From looking at the data it would seem that for all positive integers, n, the terms in the sequence P_n are divisible by 8

Use mathematical induction to prove that this is indeed the case.

[6 marks]

Question 3

(i) Explain why the product of any two consecutive integers will divide by 2.

[1 mark]

(ii) Explain why $51k(k + 1)$ is divisible by 6 for any integer k

[2 marks]

(iii) Numbers in the sequence $Z_n = 17n^3 + 103n$ are to be investigated.

n	1	2	3	4	5	6
Z_n	120	342	768	1500	2640	4290
Prime Factorisation	$2^3.3.5$	$2.3^2.19$	$2^8.3$	$2^2.3.5^3$	$2^4.3.5.11$	2.3.5.11.13

From looking at the data it would seem that for all positive integers, n, the terms in the sequence Z_n are divisible by 6.
Use mathematical induction to prove that this is indeed the case.

[6 marks]

Question 4

Numbers in the sequence $f(n) = 13^n - 6^n$ are to be investigated.

n	1	2	3	4	5	6
$f(n)$	7	133	1981	27265	363517	4780153
Prime Factorisation	7	7.19	7.283	5.7.19.41	7.11.4721	7.19.127.283

From looking at the data it would seem that for all positive integers, n, the terms in the sequence A_n are divisible by 7

(i) Show that $f(k+1) = 6f(k) + 7(13^k)$

[3 marks]

(ii) Hence, or otherwise, prove by induction that for all positive integers, n, $f(n)$ is divisible by 7

[4 marks]

Question 5

Further A-Level Examination Question from June 2019, Core 1, Q6

Prove by induction that for all positive integers n

$$f(n) = 3^{2n+4} - 2^{2n}$$

is divisible by 5

[6 marks]

Question 6

Further A-Level Examination Question from January 2018, IAL, F1, Q8 (ii)

Prove by induction, for all positive integers n,

$$f(n) = 3^{2n+3} + 40n - 27 \text{ is divisible by } 64$$

[6 marks]