Lesson 2

2.1 The 3×3 Matrix

To solve n equations with n unknowns, the working is with an $n \times n$ matrix.
In this course, the focus is upon 3 equations with 3 unknowns, and on working with a 3×3 general matrix.
The general 3×3 matrix can be expressed in two ways.

Firstly, and the method preferred by the A-Level course,

$$\mathbf{A} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

Secondly, and more easily extendable to larger matrices,

$$\begin{matrix} & & 1 & 2 & 3 \\ & 1 & & & \\ \mathbf{A} = & 2 & & & \\ & 3 & & & \end{matrix} \quad \mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{23} & a_{33} \end{pmatrix}$$

However it is expressed, for the 3×3 matrix, two questions immediately arise.

- How is its determinant worked out ?
- How is its inverse, assuming one exists, found ?

The first will be addressed in this lesson, the second in the next lesson.

2.2 The Determinant of a 3×3 Matrix

Before looking at the method of finding the determinant of a 3×3 matrix it is useful to be able to find what is termed the minor of such a matrix.

The Minor of a 3×3 Matrix

The minor of an element in a 3×3 matrix is the determinant of the 2×2 matrix that remains after the row and column containing that element are crossed out.

Example #1

Find the minor of the element 6 in the matrix $\mathbf{A} = \begin{pmatrix} 5 & -3 & 6 \\ 2 & 7 & -4 \\ 8 & 4 & 9 \end{pmatrix}$

Teaching Video : http://www.NumberWonder.co.uk/v9095/2a.mp4

Watch the video and then write out a full solution here

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[3 marks]

Spot Check

Show that, for $\mathbf{A} = \begin{pmatrix} 5 & -3 & 6 \\ 2 & 7 & -4 \\ 8 & 4 & 9 \end{pmatrix}$, $M_{21} = -51$

[3 marks]

The determinant of a 3×3 matrix is found by "expanding along a row or column". To begin with it is best to always expand along the top row. Using other rows or columns will be explored in the exercise. All, of course, arrive at the same answer !

The Determinant of a 3×3 Matrix

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

Notice that this could be written using the minors of the matrix as,

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\,M_{11} - b\,M_{12} + c\,M_{13}$$

which is easier to remember.

Example #2

Find the value of $\begin{vmatrix} 1 & 2 & 4 \\ 3 & 2 & 1 \\ -1 & 4 & 3 \end{vmatrix}$

Teaching Video : http://www.NumberWonder.co.uk/v9095/2b.mp4

Watch the video and
then write out a full
solution here

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[4 marks]

2.3 Exercise

Any solution based entirely on graphical or numerical methods is not acceptable

Marks Available : 25

Question 1

The matrix **H** is $\begin{pmatrix} -3 & 1 & 9 \\ 2 & 5 & 3 \\ -1 & 6 & 8 \end{pmatrix}$

Without using a calculator, show that **H** has a determinant of 68

[4 marks]

Question 2

Further A-Level Examination Question from June 2017, FP3, Q6 (a)

The matrix **M** is given by,

$$\mathbf{M} = \begin{pmatrix} 1 & k & 0 \\ 2 & -2 & 1 \\ -4 & 1 & -1 \end{pmatrix}, \quad k \in \mathbb{R}, \quad k \neq \frac{1}{2}$$

Show that $\mathit{det}\,\mathrm{M} = 1 - 2k$

[2 marks]

Question 3

If a matrix is described as being singular, what does this tell you about its determinant ?

[1 mark]

Question 4

Further A-Level Examination Question from June 2016, FP3, Q1

$$\mathbf{A} = \begin{pmatrix} -2 & 1 & -3 \\ k & 1 & 3 \\ 2 & -1 & k \end{pmatrix}, \text{ where } k \text{ is a constant}$$

Given that the matrix **A** is singular, find the possible values of k

[4 marks]

Question 5

The matrix $\mathbf{A} = \begin{pmatrix} 2 & -1 & 3 \\ k & 2 & 4 \\ -2 & 1 & k+3 \end{pmatrix}$, where k is a constant

Given that the determinant of **A** is 8, find the possible values of k

[5 marks]

Question 6

Every element a_{ij} in a matrix has associated with it a cofactor.

The cofactor c_{ij} is minor M_{ij} which may or may not have its sign changed.
If $i + j$ is even the cofactor is the unchanged minor, otherwise its sign is changed.
In practice, this means that the cofactors in a 3×3 matrix are the minors adjusted in accordance with the following pattern,

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

When expanding the matrix by the top row, this is where the signs in front of a, b and c came from; they were the top row of of this pattern matrix, +, –, +
That is,

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = +a\,M_{11} - b\,M_{12} + c\,M_{13}$$

Let G be the matrix, $G = \begin{pmatrix} -7 & 3 & 5 \\ 0 & 2 & 0 \\ 5 & 6 & 4 \end{pmatrix}$

Show that the determinant obtained by expansion along the top row is the same as that obtained by expansion along the middle row.

[5 marks]

Question 7

Show that, for all real values of x, the matrix $\begin{pmatrix} 2 & -2 & 4 \\ 3 & x & -2 \\ -1 & 3 & x \end{pmatrix}$ is non singular

[4 marks]