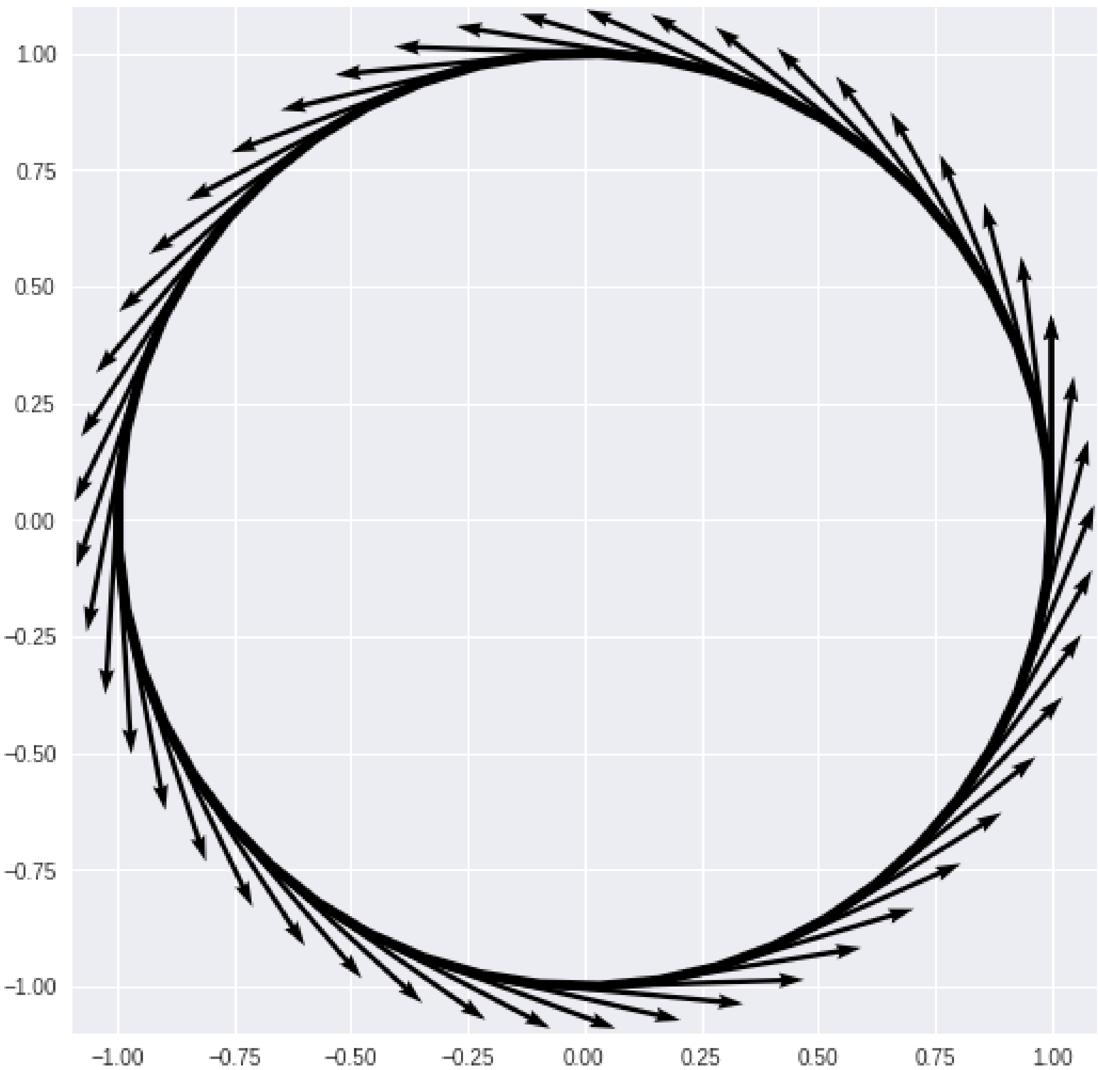
Further Pure A-Level Mathematics
Compulsory Course Component
Core 1

VECTORS

III



Vectors III

Lesson 1

Further A-Level Pure Mathematics
Vectors III : Core 1

1.1 Introduction

Vectors provide us with a powerful means of working with straight lines in two and three dimensions, extendable to higher dimensions if required. Unlike the approach based on trigonometry, vector methods do not become more intricate as the number of dimensions is increased.

To illuminate the vector approach, a two dimensional problem will be tackled first using familiar GCSE methods, then the Further A-Level vectors approach.

1.2 The GCSE Method

Consider the two straight lines;

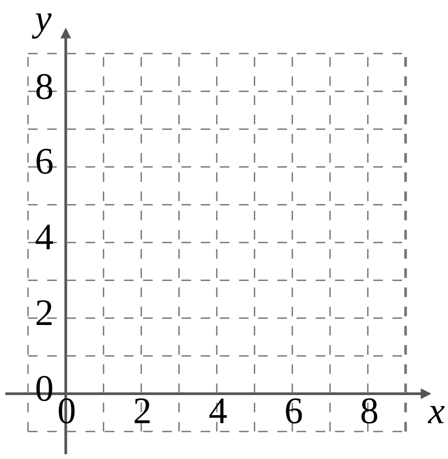
$$y = 2x + 1 \qquad \text{and} \qquad y = \frac{1}{2}x + 4$$

(i) On the grid below, sketch the two lines.

(ii) Write down the point of intersection.

(iii) Show how to find this point of intersection using simple algebra.

(iv) Find θ, the acute angle in degrees, between the two lines using trigonometry.



[6 marks]

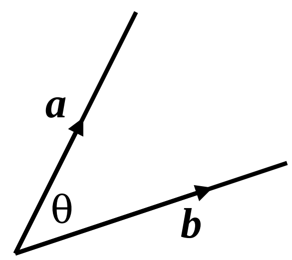
1.3 The Vector Method

[6 marks]

1.4 Angles and Vectors

The Scalar Product (Also called **The Dot Product**)

By definition : $\boldsymbol{a} \bullet \boldsymbol{b} = |\boldsymbol{a}|\ |\boldsymbol{b}|\ \cos\theta$



Where $|\boldsymbol{a}|$ and $|\boldsymbol{b}|$ are the magnitudes of vectors ***a*** and ***b***

θ is the angle between vectors ***a*** and ***b***

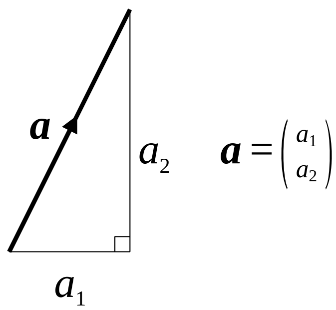
Whenever angles are involved in a problem involving vectors, use the scalar product with the **DIRECTION PARTS** of the vector equations of the lines.

The scalar product will be studied in Lesson 3, but for now work with;

In two dimensions;

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \bullet \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

$$a_1b_1 + a_2b_2 = \sqrt{a_1^2 + a_2^2}\ \sqrt{b_1^2 + b_2^2}\ \cos\theta$$



In three dimensions;

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \bullet \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

$$a_1b_1 + a_2b_2 + a_3b_3 = \sqrt{a_1^2 + a_2^2 + a_3^2}\ \sqrt{b_1^2 + b_2^2 + b_3^2}\ \cos\theta$$

1.5 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 60

Question 1

Consider the two straight lines;

$$\boldsymbol{r_1} = \begin{pmatrix} -4 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 5 \\ 2 \end{pmatrix}$$

and $$\boldsymbol{r_2} = \begin{pmatrix} 3 \\ -8 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ 7 \end{pmatrix}$$

(i) Write down the direction part of $\boldsymbol{r_1}$

[1 mark]

(ii) Use the scalar product to show that;

$$cos\,\theta = \frac{1}{\sqrt{2}}$$

[3 marks]

(iii) Hence determine, in degrees, the angle between the two lines.

[1 mark]

Question 2

Consider the two straight lines;

$$\boldsymbol{r_1} = \begin{pmatrix} -7 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 3 \end{pmatrix}$$

and $$\boldsymbol{r_2} = \begin{pmatrix} -5 \\ 8 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ 3 \end{pmatrix}$$

(i) Write down the direction part of $\boldsymbol{r_2}$

[1 mark]

(ii) Use the dot product to show that;

$$\cos\theta = \frac{1}{\sqrt{10}}$$

[3 marks]

(iii) Hence determine the angle between the two lines.

[1 mark]

Question 3

Consider the two straight lines;

$$\boldsymbol{r_1} = \begin{pmatrix} -4 \\ -11 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\boldsymbol{r_2} = \begin{pmatrix} -8 \\ 11 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

Use vector methods to determine the point of intersection of the two lines.

[4 marks]

Question 4

Consider the two straight lines;

$$\boldsymbol{r_1} = \begin{pmatrix} -15 \\ -4 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

$$\boldsymbol{r_2} = \begin{pmatrix} 15 \\ 17 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

Use vector methods to determine the point of intersection of the two lines.

[4 marks]

Question 5

Consider the two straight lines;

$$\boldsymbol{r_1} = \begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 4 \\ 12 \end{pmatrix}$$

and $$\boldsymbol{r_2} = \begin{pmatrix} 4 \\ -1 \\ -3 \end{pmatrix} + \mu \begin{pmatrix} -3 \\ -4 \\ 5 \end{pmatrix}$$

(i) Write down the direction part of $\boldsymbol{r_1}$

[1 mark]

(ii) Use the scalar product to show that;

$$cos\,\theta = \frac{7\sqrt{2}}{26}$$

[3 marks]

(iii) Hence determine, in degrees, the angle between the two lines.

[1 mark]

Question 6

A, B and C are the points;

$$A(2,\ 3,\ 7) \qquad B(8,\ 5,\ 9) \qquad C(3,\ 11,\ 4)$$

(i) Show that $\overrightarrow{AB} = \begin{pmatrix} 6 \\ 2 \\ 2 \end{pmatrix}$

[1 mark]

(ii) Determine $\overrightarrow{AC}$

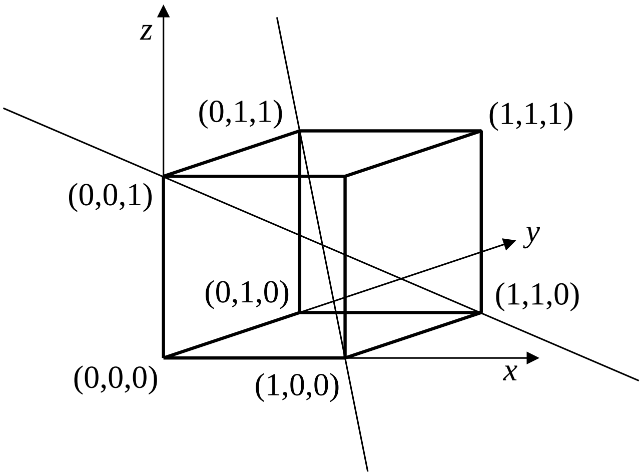
[1 mark]

(iii) Find $\angle$BAC by considering the scalar product, $\overrightarrow{AB} \bullet \overrightarrow{AC}$

[3 marks]

Question 7

A cube has the eight vertices shown in the following diagram,



(i) One long diagonal passes through the points *A* (1,0,0) and *B* (0,1,1)
Explain why this diagonal line has the vector equation;

$$\boldsymbol{r}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

[1 mark]

(ii) Another long diagonal passes through the points *C* (1,1,0) and *D* (0,0,1)
Determine the vector equation of this diagonal line.

[2 mark]

(iii) Use the scalar product to determine the acute angle between the two long diagonals. Give your answer in degrees, accurate to one decimal place.

[3 marks]

Question 8
C4 Examination Question from January 2010, Q4

The line l_1 has vector equation

$$\mathbf{r_1} = \begin{pmatrix} -6 \\ 4 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ -1 \\ 3 \end{pmatrix}$$

and the line l_2 has vector equation

$$\mathbf{r_2} = \begin{pmatrix} -6 \\ 4 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ -4 \\ 1 \end{pmatrix}$$

where λ and μ are parameters.

The lines l_1 and l_2 intersect at the point A and the acute angle between l_1 and l_2 is θ

(a) Write down the coordinates of A

[1 mark]

(b) Find the value of $\cos\theta$

[3 marks]

The point X lies on l_1 where $\lambda = 4$

(c) Find the coordinates of X

[1 mark]

(d) Find the vector $\overrightarrow{AX}$

[2 marks]

(e) Hence, or otherwise, show that

$$\left|\overrightarrow{AX}\right| = 4\sqrt{26}$$

[2 marks]

The point Y lies on l_2.
Given that the vector $\overrightarrow{YX}$ is perpendicular to l_1

(f) find the length of AY, giving your answer to 3 significant figures

[3 marks]

Question 9

The lines l_1 and l_2 have equations;

$$l_1 \quad : \quad \boldsymbol{r_1} = \boldsymbol{i} - \boldsymbol{j} - 5\boldsymbol{k} + \lambda(2\boldsymbol{i} + 3\boldsymbol{j} + 4\boldsymbol{k})$$

$$l_2 \quad : \quad \boldsymbol{r_2} = 16\boldsymbol{i} + 5\boldsymbol{j} - 2\boldsymbol{k} + \mu(7\boldsymbol{i} - 6\boldsymbol{j} - 10\boldsymbol{k})$$

Find the acute angle, to the nearest 0.1°, between l_1 and l_2

[4 marks]

Question 10

A straight line has equation;

$$\boldsymbol{r} = \begin{pmatrix} 3 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 5 \end{pmatrix}$$

(i) Which part of the equation is the direction part ?

[1 mark]

(ii) Show that the point (13, 23) is on the line.

[2 marks]

(iii) Where does this line cross the x-axis ?

[2 marks]

Question 11

A straight line has equation;

$$\boldsymbol{r} = \begin{pmatrix} 3 \\ 7 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix}$$

(i) Which part of the equation is the direction part ?

[1 mark]

(ii) Show that the point (19, 47, 11) is on the line.

(iii) Show that this line does not cross the *x*-axis.