3.1 Cubic and Roots

Having taken a fresh look at quadratic equations through a study of their roots, the focus can now shift to cubic equations. Thanks to Gerolamo Cardano it is known that there is, in principle, a formula to solve cubics. Alas, it has a square root nested inside a cube root which makes manipulating it intricate. Fortunately, the algebra can be skirted around by a more thoughtful approach. To illustrate the idea, it will first be applied it to the generalised quadratic equation.

3.2 A Proof Rewritten

The Roots of a Quadratic

If α and β are the roots of the equation $a\,x^2 + bx + c = 0 \qquad a,\ b,\ c \in \mathbb{C}$

then, $\qquad \alpha + \beta = -\dfrac{b}{a} \qquad\qquad \alpha\beta = \dfrac{c}{a}$

Proof

For a general quadratic, with roots α and β,

$$\begin{aligned} a\,x^2 + bx + c &= a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) \\ &= a\left((x - \alpha)(x - \beta)\right) \\ &= a\left(x^2 - \alpha x - \beta x + \alpha\beta\right) \\ &= a\left(x^2 - (\alpha + \beta)x + \alpha\beta\right) \end{aligned}$$

From which can be seen that,

$$\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) = \left(x^2 - (\alpha + \beta)x + \alpha\beta\right)$$

And, by matching coefficients of x, the deduction made that,

$$\alpha + \beta = -\frac{b}{a} \quad \text{and} \quad \alpha\beta = \frac{c}{a} \qquad\qquad \square$$

The key feature of this proof, in comparison with that presented in Lesson 1, is that no use is made of the formula,

$$x = \frac{-b \pm \sqrt{b^2 + 4ac}}{2a}$$

3.3 A Similar Proof For Cubics

The Roots of a Cubic

If α, β and γ are roots of $ax^3 + bx^2 + cx + d = 0$ $\quad a, b, c, d \in \mathbb{C}$

then, $\quad \alpha + \beta + \gamma = -\frac{b}{a} \qquad \alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a} \qquad \alpha\beta\gamma = -\frac{d}{a}$

Proof

For a general cubic, with roots α, β and γ,

$$\begin{aligned} ax^3 + bx^2 + cx + d &= a\left(x^3 + \frac{b}{a}x^2 + \frac{c}{a}x + \frac{d}{a}\right) \\ &= a((x-\alpha)(x-\beta)(x-\gamma)) \\ &= a((x^2 - \alpha x - \beta x + \alpha\beta)(x-\gamma)) \\ &= a((x^2 - \alpha x - \beta x + \alpha\beta)x - (x^2 - \alpha x - \beta x + \alpha\beta)\gamma) \\ &= a(x^3 - \alpha x^2 - \beta x^2 + \alpha\beta x - \gamma x^2 + \alpha\gamma x + \beta\gamma x - \alpha\beta\gamma) \\ &= a(x^3 - (\alpha + \beta + \gamma)x^2 + (\alpha\beta + \beta\gamma + \gamma\alpha)x - \alpha\beta\gamma) \end{aligned}$$

from which can be seen that,

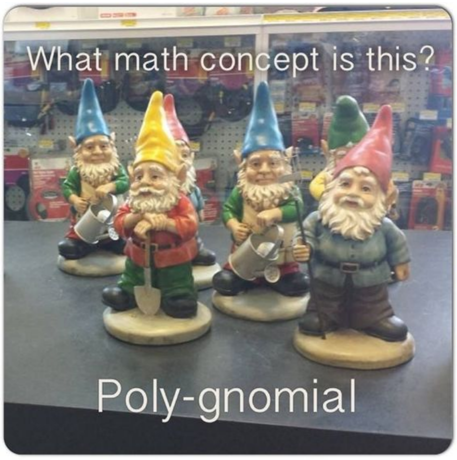
$$\left(x^3 + \frac{b}{a}x^2 + \frac{c}{a}x + \frac{d}{a}\right) = (x^3 - (\alpha + \beta + \gamma)x^2 + (\alpha\beta + \beta\gamma + \gamma\alpha)x - \alpha\beta\gamma)$$

and, by matching coefficients of x, the deduction made that,

$$\alpha + \beta + \lambda = -\frac{b}{a}, \quad \alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a} \quad \text{and } \alpha\beta\gamma = -\frac{d}{a} \qquad \square$$

Notice that, as with the quadratic,

- The sum of the roots is $\left(-\frac{b}{a}\right)$
- The product of all possible pairs of roots is $\left(\frac{c}{a}\right)$



3.4 Example

The roots of the cubic equation $2x^3 + 5x^2 - 2x + 3 = 0$ are α, β and γ.

Without solving the equation, find the value of $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$

Teaching Video : http://www.NumberWonder.co.uk/v9093/3.mp4

Watch the teaching video and then write out a solution to the question.

☞

[4 marks]

3.5 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable.
Make the method used clear.
Marks available : 40

Question 1

The roots of the equation $4x^3 - 3x^2 - x + 6 = 0$ are α, β and γ

Without solving the equation, find the roots of

(i) $\alpha + \beta + \gamma$ **(ii)** $\alpha\beta + \beta\gamma + \gamma\alpha$

[1, 1 mark]

(iii) $\alpha^2\beta^2\gamma^2$ **(iv)** $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$

[1, 2 marks]

Question 2

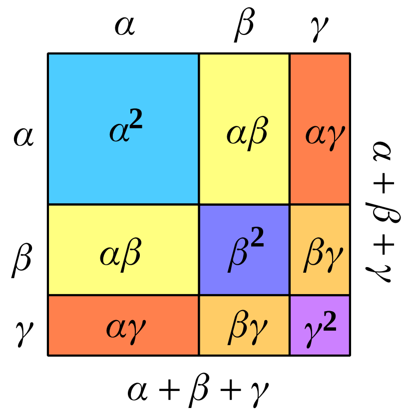
The roots of the cubic equation $ax^3 + bx^2 + cx + d = 0$ are,

$$\alpha = 8, \quad \beta = 9 \quad \text{and} \quad \gamma = -10$$

Find integer values for a, b, c and d

[2 marks]

Question 3



(i) With the aid of the diagram expand the brackets of $(\alpha + \beta + \gamma)^2$

[1 mark]

The roots of the equation $2x^3 + 4x^2 + 7x + 1 = 0$ are α, β and γ

Without solving the equation, write down the values of,

(ii) $\alpha + \beta + \gamma$

[1 mark]

(iii) $\alpha\beta + \beta\gamma + \gamma\alpha$

[1 mark]

(iv) $\alpha\beta\gamma$

[1 mark]

(v) $\alpha^2 + \beta^2 + \gamma^2$

[2 marks]

Question 4

Further A-Level Examination Question from June 2017 SAM, Core 2, Q1

The roots of the equation $x^3 - 8x^2 + 28x - 32 = 0$ are α, β and γ

Without solving the equation, find the value of

(i) $\dfrac{1}{\alpha} + \dfrac{1}{\beta} + \dfrac{1}{\gamma}$

[3 marks]

(ii) $(\alpha + 2)(\beta + 2)(\gamma + 2)$

[3 marks]

(iii) $\alpha^2 + \beta^2 + \gamma^2$

[2 marks]

Question 5

Our sequence of questions exploring Cardano's extraordinary achievement in finding and using a formula to solve cubic equations continues by looking at,

$$x^3 - 252x + 1296 = 0$$

As you will discover, this example involves complex numbers.
Here is a recap of the theory to be applied;

Roots of a Cubic

Given a depressed cubic of the form

$$t^3 + pt + q = 0 \qquad\qquad p,\ q \in \mathbb{C}$$

where p and q are not both zero, and $4p^3 + 27q^2 \neq 0$, calculate,

$$C = \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$$

A root of the cubic is then given by,

$$\alpha = C - \frac{p}{3C}$$

Polynomial division can then be used to find the remaining roots.

To help with a tricky step, there is a part (i) which is useful when tackling the main problem.

(i) Expand the brackets,

$$\left(6 + 4\sqrt{3}\,\mathrm{i}\right)^3 \quad \text{where } \mathrm{i} = \sqrt{-1} \text{ such that } \mathrm{i}^2 = -1$$

giving your answer in the form $a\sqrt{3}\,\mathrm{i} + b$ for integer a and b

[3 marks]

(ii) Determine using the method of Cardano, the real root of the cubic,

$$x^3 - 252x + 1296 = 0$$

[3 marks]

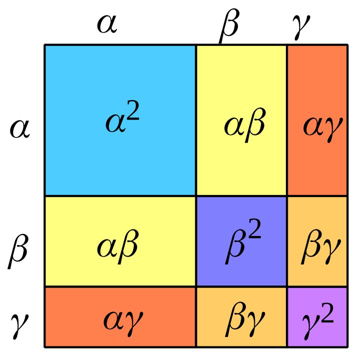
(iii) Using polynomial division with your part (ii) answer, find all three roots, two of which are a complex conjugate pair, of the depressed cubic,

$$x^3 - 252x + 1296 = 0$$

[2 marks]

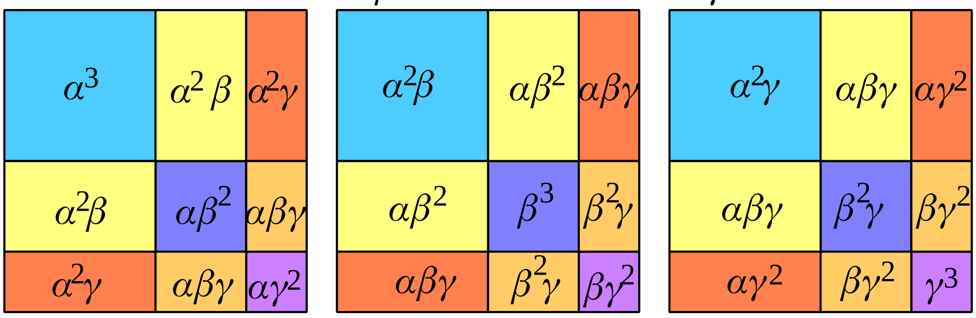
Question 6

$(\alpha+\beta+\gamma)^2 =$



$(\alpha+\beta+\gamma)^3 =$

$\alpha(\alpha+\beta+\gamma)^2 \quad + \quad \beta(\alpha+\beta+\gamma)^2 \quad + \quad \gamma(\alpha+\beta+\gamma)^2$



(i) In question 3 a square with sides of length $\alpha + \beta + \gamma$ was of assistance in expanding the brackets of $(\alpha + \beta + \gamma)^2$.

For $(\alpha + \beta + \gamma)^3$ a cube can be used in which case the three layers of the cube give cuboids with volumes as shown above.

By using the diagrams or otherwise, expand the brackets of $(\alpha + \beta + \gamma)^3$

[1 mark]

(ii) Manipulate your part (i) answer into a form that will allow you to express $\alpha^3 + \beta^3 + \gamma^3$ in terms of $\alpha + \beta + \gamma$, $\alpha\beta + \beta\gamma + \gamma\alpha$ and $\alpha\beta\gamma$

[2 marks]

Question 7

Further A-Level Examination Question from June 2019, Core 2, Q2

The roots of the equation $x^3 - 2x^2 + 4x - 5 = 0$ are p, q and r

Without solving the equation, find the value of

(i) $\frac{2}{p} + \frac{2}{q} + \frac{2}{r}$

[3 marks]

(ii) $(p - 4)(q - 4)(r - 4)$

[3 marks]

(iii) $p^3 + q^3 + r^3$

[2 marks]