7.1 Animated Conic Sections

Conic Sections are formed when a double cone is sliced in a variety of ways. The possible slices obtained can be circular, elliptical, parabolic or hyperbolic in shape. The following two minute video is an animation of the slicing process along with the corresponding graph. It shows that a few other surprising results can occur; watch out for the graph being a single point or a single straight line. The most interesting of these “surprise results” is the double straight line.

Teaching Video : http://www.NumberWonder.co.uk/v9091/7.mp4

7.2 The Double Straight Line (Hyperbola)

The recent GCSE examination papers have included some tough questions on conic sections, tackled from an algebraic point of view. As an example of the sort of question being asked, consider the following;

The Question

Use algebra to solve the following equations simultaneously,

$$x^2 - 4xy + 3y^2 = 0$$

$$y = 2x - 5$$

Give It A Go

Your job now is to tackle this on a scrap piece of paper using algebra.
The important thing to notice is that the y in

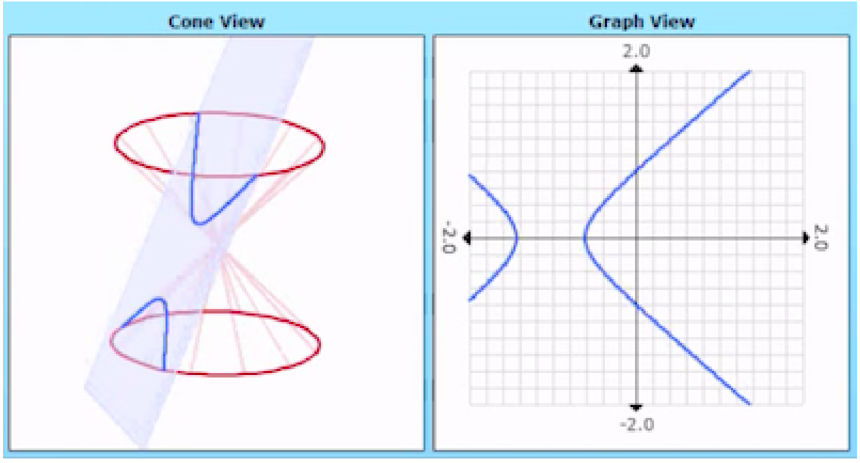
$$x^2 - 4xy + 3y^2 = 0$$

is in two places.
Each of those needs to be replaced with $2x - 5$ from the other equation.

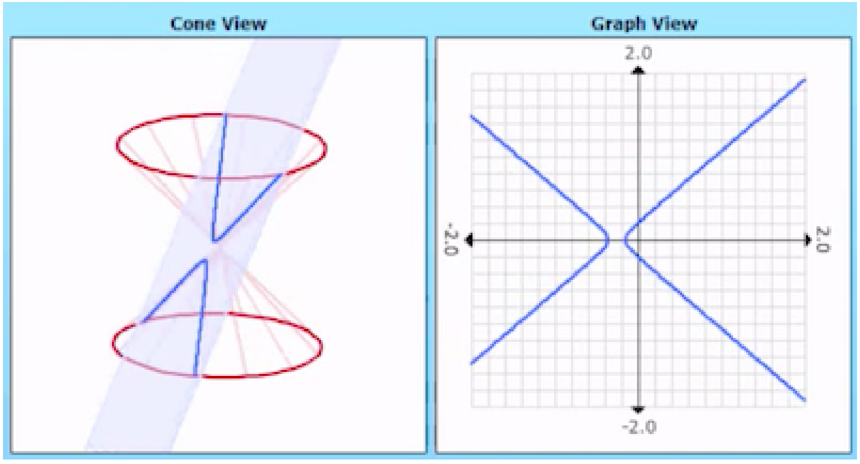
Over the page, what the graph of $x^2 - 4xy + 3y^2 = 0$ looks like will be revealed and a full solution to the question will also be provided.

The Graph
The animation video showed how a hyperbola can degenerate into a graph with two straight lines.

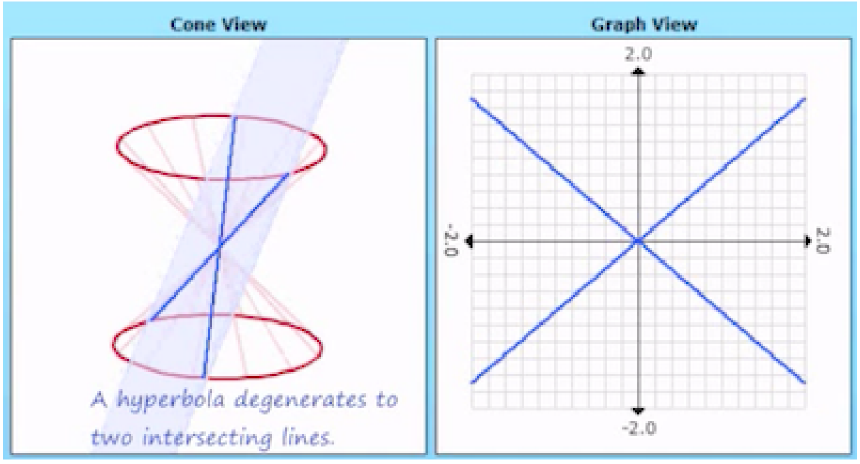
Frame 1



Frame 2



Frame 3



The equation of this example

$$x^2 - 4xy + 3y^2 = 0$$

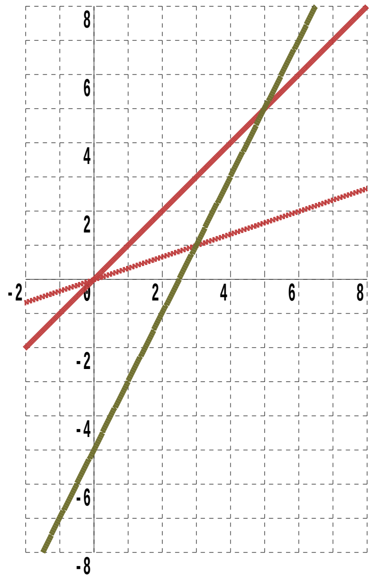
is such a double straight line graph.

The “more is less” hyperbola that you know and love,

$$y = \frac{k}{x} \text{ for } k \text{ a constant}$$

has been rotated and squashed. If anything, it's amazing that after such a mangling the resulting equation is not far more complicated.

Here is an accurately plotted graph of the double straight line $x^2 - 4xy + 3y^2 = 0$ in pink, with the straight line $y = 2x - 5$ in green.



From this graph you can see the two points of intersection.

They are (3, 1) and (5, 5) : Did you get them from your algebra ?

If you need help sorting out the algebra, take a look at section 14.4 which presents a detailed worked solution. (It's after the 14.3 Exercise)

7.3 Exercise

Question 1

GCSE Examination Question from January 2020, Paper 2H, Q22

The line with equation $y = x + 2$ intersects the curve with equation $x^2 + y^2 - 2y = 24$ at the points *A* and *B*.

Find the coordinates of *A* and *B*.

Show clear algebraic working.

[5 marks]

Question 2

GCSE Examination Question from June 2018, Paper 2HR, Q18

Solve the simultaneous equations,

$$2x^2 + 3y^2 = 14$$

$$x = 2y - 3$$

Show clear algebraic working,

[5 marks]

Question 3

GCSE Question from May 2018, Paper 1H, Q11(b)

Solve $3x^2 + 6x - 5 = 0$

Show your working clearly

Give your solutions correct to 3 significant figures.

[3 marks]

Question 4

GCSE Examination Question from June 2019, Paper 2H, Q20

The equation of the line **L** is $y = 9 - x$

The equation of the curve C is $x^2 - 3xy + 2y^2 = 0$

L and **C** intersect at two points.

Find the coordinates of those two points.

Show clear algebraic working.

[5 marks]

7.4 The 7.2 Example's Worked Solution

The Question

Use algebra to solve the following equations simultaneously,

$$x^2 - 4xy + 3y^2 = 0$$

$$y = 2x - 5$$

The Answer

$$x^2 - 4x(2x - 5) + 3[(2x - 5)^2] = 0$$

$$x^2 - 8x^2 + 20x + 3[4x^2 - 20x + 25] = 0$$

$$x^2 - 8x^2 + 20x + 12x^2 - 60x + 75 = 0$$

$$5x^2 - 40x + 75 = 0$$

Divide through by 5

$$x^2 - 8x + 15 = 0$$

$$(x - 3)(x - 5) = 0$$

Either $x - 3 = 0$ or $x - 5 = 0$

$x = 3$ or $x = 5$

Use $y = 2x - 5$ to get the y-part of the coordinates

Answer : Points of intersection are (3, 1) or (5, 5)