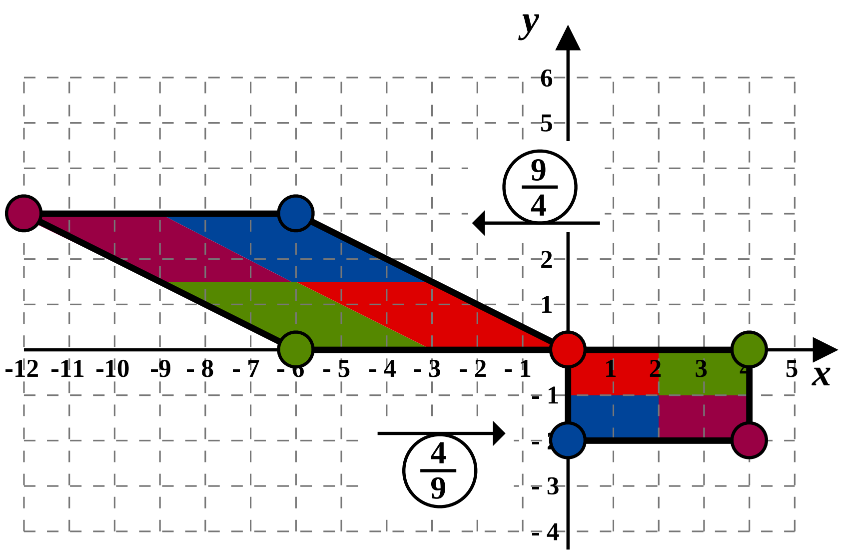
9.1 The Undo Matrix

Let the matrix **T** be $\begin{pmatrix} -1.5 & 3 \\ 0 & -1.5 \end{pmatrix}$

The transformation represented by this matrix is applied to the rectangle described by $\mathbf{R} = \begin{pmatrix} 0 & 4 & 4 & 0 \\ 0 & 0 & -2 & -2 \end{pmatrix}$

TR	$\begin{pmatrix} 0 & 4 & 4 & 0 \\ 0 & 0 & -2 & -2 \end{pmatrix}$
$\begin{pmatrix} -1.5 & 3 \\ 0 & -1.5 \end{pmatrix}$	$\begin{pmatrix} 0 & -6 & -12 & -6 \\ 0 & 0 & 3 & 3 \end{pmatrix}$



The graph shows the original rectangle and its image, a parallelogram.

Also shown is the area scale factor of the transformation, the $\frac{9}{4}$.

This was calculated from,

$$\begin{aligned} det\ \mathbf{T} &= (-1.5) \times (-1.5) - 3 \times 0 \\ &= 2.25 \end{aligned}$$

Writing this as $\frac{9}{4}$ is helpful as it's then obvious the inverse area scale factor is $\frac{4}{9}$.

In starting to think about the inverse transformation that will move the parallelogram back to the rectangle, one would expect $\frac{1}{det\ \mathbf{T}}$ to be involved, given the connection between determinants and area scale factor. This insight also illuminates why a determinant of zero is a problem; it's the familiar issue of division by zero not being defined in mathematics emerging once again. Singular matrices can have no inverse.

9.2 Inverting a 2 × 2 Matrix

Given that we have been using matrices to move points, the inverse matrix will be that which moves the points back from whence they came.

The Inverse of a Matrix

In general, the inverse of a non-singular matrix **M** is the matrix $\mathbf{M}^{-1}$ such that

$$\mathbf{M}\mathbf{M}^{-1} = \mathbf{M}^{-1}\mathbf{M} = \mathbf{I}$$

In particular, if $\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $\mathbf{M}^{-1} = \dfrac{1}{det\,\mathbf{M}} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

For the matrix $\mathbf{T} = \begin{pmatrix} -1.5 & 3 \\ 0 & -1.5 \end{pmatrix}$ where $det\,\mathbf{T} = \dfrac{9}{4}$, we have $\dfrac{1}{det\,\mathbf{T}} = \dfrac{4}{9}$

Thus,

$$\begin{aligned} \mathbf{T}^{-1} &= \frac{4}{9}\begin{pmatrix} -1.5 & -3 \\ 0 & -1.5 \end{pmatrix} \\ &= -\frac{2}{3}\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

Now to show that this inverse matrix moves the parallelogram back to the square.

$$\begin{aligned} -\frac{2}{3}\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -6 & -12 & -6 \\ 0 & 0 & 3 & 3 \end{pmatrix} &= -\frac{2}{3}\begin{pmatrix} 0 & -6 & -6 & 0 \\ 0 & 0 & -3 & 3 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 4 & 4 & 0 \\ 0 & 0 & 2 & -2 \end{pmatrix} \end{aligned}$$

□

9.3 Spot Check

Find the inverse of each of the following matrices.

Having done so, check your answers with mine, to be found in 9.5 after the exercise.

For **C**, if you wish, you can deal with the third separately, inverting it to a 3.

$\mathbf{A} = \begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix}$ $\quad\mathbf{B} = \begin{pmatrix} 4 & 6 \\ 10 & 16 \end{pmatrix}$ $\quad\mathbf{C} = \dfrac{1}{3}\begin{pmatrix} 6 & -3 \\ -7 & 4 \end{pmatrix}$

9.4 Exercise

Question 1

Further A-Level Examination Question from January 2013, FP1, Q6

$$\mathbf{X} = \begin{pmatrix} 1 & a \\ 3 & 2 \end{pmatrix}$$

(a) Find the value of a for which the matrix X is singular

[2 marks]

$$\mathbf{Y} = \begin{pmatrix} 1 & -1 \\ 3 & 2 \end{pmatrix}$$

(b) Find $\mathbf{Y}^{-1}$

[2 marks]

The transformation represented by **Y** maps the point *A* onto the point *B*
Given that *B* has coordinates $(1 - \lambda,\ 7\lambda - 2)$ where λ is a constant,

(c) find, in terms of λ, the coordinates of point *A*

[4 marks]

Question 2

Further A-Level Examination Question from January 2011, FP1, Q8

$$\mathbf{A} = \begin{pmatrix} 2 & -2 \\ -1 & 3 \end{pmatrix}$$

(a) Find *det* **A**

[1 mark]

(b) Find $\mathbf{A}^{-1}$

[2 marks]

The triangle *R* is transformed to the triangle *S* by the matrix **A**
Given that the area of triangle *S* is 72 square units,

(c) find the area of triangle *R*

[2 marks]

The triangle *S* has vertices at the points (0, 4), (8, 16) and (12, 4).

(d) Find the coordinates of the vertices of *R*

[4 marks]

Question 3

Further A-Level Examination Question from January 2010, Q5

$$\mathbf{A} = \begin{pmatrix} a & -5 \\ 2 & a+4 \end{pmatrix}, \quad \text{where } a \text{ is real}$$

(a) Find *det* **A** in terms of *a*

[2 marks]

(b) Show that the matrix **A** is non-singular for all values of *a*

[3 marks]

Given that $a = 0$,

(c) find $\mathbf{A}^{-1}$

[3 marks]

Question 4

Further A-Level Examination Question from June 2015, Q7

(i) $\mathbf{A} = \begin{pmatrix} 5k & 3k - 1 \\ -3 & k + 1 \end{pmatrix}$, where k is a real constant.

Given that **A** is a singular matrix, find the possible values of k.

[4 marks]

(ii) $\mathbf{B} = \begin{pmatrix} 10 & 5 \\ -3 & 3 \end{pmatrix}$

A triangle T is transformed onto a triangle T' by the transformation represented by the matrix **B**. The vertices of triangle T' have coordinates $(0, 0)$, $(-20, 6)$ and $(10c, 6c)$, where c is a positive constant.

The area of triangle T' is 135 square units.

(a) Find the matrix $\mathbf{B}^{-1}$

[2 marks]

(b) Find the coordinates of the vertices of the triangle T in terms of c where necessary.

[3 marks]

(c) Find the value of c

[3 marks]

9.5 Answers to the 9.3 Spot Check

$$\mathbf{A}^{-1} = \frac{1}{3 \times 2 - 1 \times 5}\begin{pmatrix} 2 & -1 \\ -5 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & -1 \\ -5 & 3 \end{pmatrix}$$

$$\mathbf{B}^{-1} = \frac{1}{4 \times 16 - 6 \times 10}\begin{pmatrix} 16 & -6 \\ -10 & 4 \end{pmatrix}$$

$$= \frac{1}{4}\begin{pmatrix} 16 & -6 \\ -10 & 4 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 8 & -3 \\ -5 & 2 \end{pmatrix}$$

$$\mathbf{C}^{-1} = \frac{3}{6 \times 4 - (-3) \times (-7)}\begin{pmatrix} 4 & 3 \\ 7 & 6 \end{pmatrix}$$

$$= \frac{3}{3}\begin{pmatrix} 4 & 3 \\ 7 & 6 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & 3 \\ 7 & 6 \end{pmatrix}$$