6.1 Generalised Rotation

The rotation matrices looked at previously are special cases of a more general result,

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Rotation through angle θ about (0, 0)

$$\mathbf{R}_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

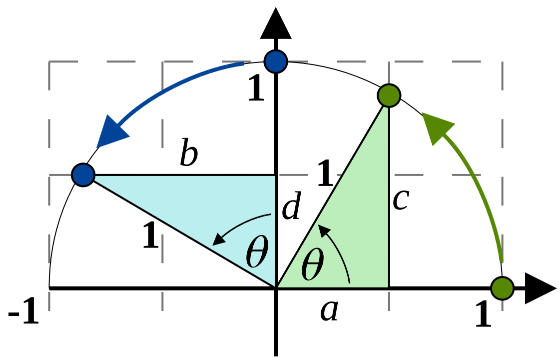
Proof

Let the matrix that causes rotation of $\theta°$ about the origin be $\mathbf{R}_\theta = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

Consider what this matrix does to the unit square, $\mathbf{U} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$

$\mathbf{R}_\theta\mathbf{U}$	$\begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$
$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$	$\begin{pmatrix} 0 & a & a+b & b \\ 0 & c & c+d & d \end{pmatrix}$

Thus, under the rotation of $\theta°$ $(1, 0) \rightarrow (a, c)$ and $(0, 1) \rightarrow (b, d)$



From the diagram, the green triangle has hypotenuse 1, opposite c and adjacent a
Applying SOH CAH TOA in this green triangle gives $a = \cos\theta$ and $c = \sin\theta$

The displacement vector to the rotated green point is thus $\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$.

The green point has coordinates ($\cos\theta, \sin\theta$).
The blue and green triangles are congruent, with lengths a and c equal to lengths d and b respectively; the displacement vector to the rotated blue point is $\begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$.

The blue point has coordinates ($-\sin\theta$, $\cos\theta$).

Thus $\mathbf{R}_\theta = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ □

Exercise 6.2

Question 1

(i) By comparing the matrix $\mathbf{Y} = \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$ to $\mathbf{R}_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

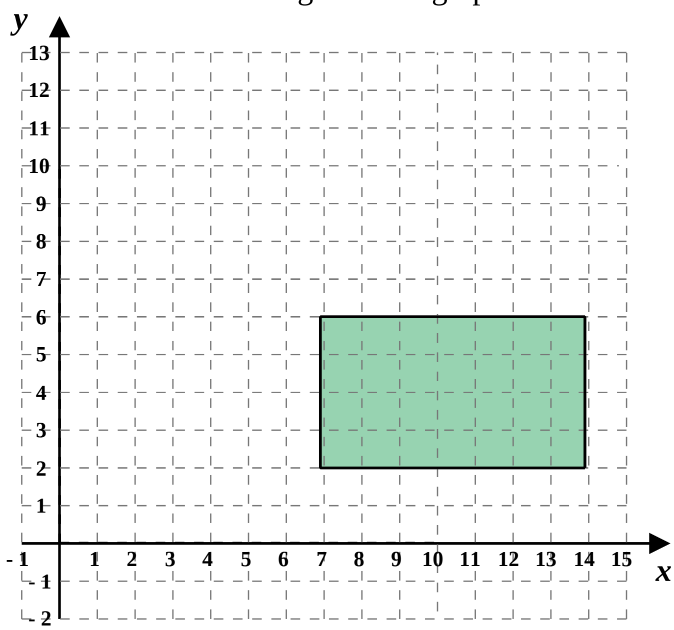
state the transformation represented by matrix **Y**.

(ii) A rectangle is represented by the multipoint matrix,

$$\mathbf{M} = \begin{pmatrix} 4\sqrt{3} & 8\sqrt{3} & 8\sqrt{3} & 4\sqrt{3} \\ 2 & 2 & 6 & 6 \end{pmatrix}$$

Apply the transformation represented by matrix **Y** to the rectangle.
Give the exact coordinates of the transformed rectangle.

(iii) Plot the transformed rectangle on the graph below.



Question 2

Further A-Level Examination Question from February 2010, FP1, Q9

$$\mathbf{M} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

(a) Describe fully the geometrical transformation represented by the matrix **M**

[2 marks]

The transformation represented by **M** maps the point *A* with coordinates (*p*, *q*) onto the point *B* with coordinates $\left(3\sqrt{2},\ 4\sqrt{2} \right)$

(b) Find the value of *p* and the value of *q*

[4 marks]

(c) Find, in its simplest surd form, the length *OA*, where *O* is the origin.

[2 marks]

(d) Find $\mathbf{M}^2$

[2 marks]

The point *B* is mapped onto the point *C* by the transformation represented by $\mathbf{M}^2$

(e) Find the coordinates of *C*

[2 marks]

Question 3

Further A-Level Examination Question from May 2016, FP1, Q6

$$\mathbf{P} = \begin{pmatrix} \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{pmatrix}$$

(a) Describe fully the single geometrical transformation U represented by the matrix **P**

[2 marks]

The transformation U maps the point A with coordinates (p, q) onto the point B, with coordinates $(6\sqrt{2},\ 3\sqrt{2})$

(b) Find the value of p and the value of q

[3 marks]

The transformation V, represented by the 2×2 matrix **Q**, is a reflection in the line with equation $y = x$.

(c) Write down the matrix **Q**

[1 mark]

The transformation U followed by the transformation V is the transformation T. The transformation T is represented by the matrix **R**

(d) Find the matrix **R**

[3 marks]

(e) Deduce that the transformation T is self-inverse

[2 marks]

Question 4

Further A-Level Examination Question from June 2014, FP1, Q7

(i) In each of the following cases, find a 2×2 matrix that represents

(a) a reflection in the line $y = -x$

(b) a rotation of 135° anticlockwise about (0, 0)

(c) a reflection in the line $y = -x$ followed by a rotation of 135° anticlockwise about (0, 0)

[4 marks]

(ii) The triangle T has vertices at the points (1, k) (3, 0) and (11, 0) where k is a constant,

The triangle T is transformed onto the triangle T' by the matrix

$$\begin{pmatrix} 6 & -2 \\ 1 & 2 \end{pmatrix}$$

Given that the area of triangle T' is 364 square units, find the value of k

[6 marks]

Question 5

(i) Given that **P** = $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

show algebraically that $\mathbf{P}^2 = \begin{pmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix}$

(ii) Interpret this result geometrically