Lesson 2

Further A-Level Pure Mathematics
Matrix Transformations : Core 1

2.1 Point Transformations

A square matrix of size 2×2 can be thought of as a transformation that moves a point on a two dimensional surface to a different location on that surface. This is an application of what is termed "matrix multiplication". This is a completely new type of mathematical operation, and the word "multiplication" is given a new meaning in the context of manipulating matrices. The rule for performing matrix multiplication with a 2×2 matrix on a point is now given;

Point Transformation by a Matrix (Two Dimensions)

The point $(x,\ y)$ is written $\begin{pmatrix} x \\ y \end{pmatrix}$ and placed right of the transforming matrix.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}$$

The point $(x,\ y)$ has been transformed to the point $(ax + by,\ cx + dy)$

2.2 Four Practice Questions

Use the above rule to transform the given point.
Once done, check your answers with those over the page

(i) $\begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 3 \end{pmatrix}$

$(1, 3) \rightarrow (\quad ,\quad)$

(ii) $\begin{pmatrix} 4 & -1 \\ 6 & 2 \end{pmatrix}\begin{pmatrix} 3 \\ 5 \end{pmatrix}$

$(3, 5) \rightarrow (\quad ,\quad)$

(iii) $\begin{pmatrix} 7 & 4 \\ 5 & 3 \end{pmatrix}\begin{pmatrix} 2 \\ -3 \end{pmatrix}$

$(2, -3) \rightarrow (\quad ,\quad)$

(iv) $\begin{pmatrix} 4 & -1 \\ -8 & 5 \end{pmatrix}\begin{pmatrix} -3 \\ 5 \end{pmatrix}$

$(-3, 5) \rightarrow (\quad ,\quad)$

2.3 Practice Question's Answers

(i) $\begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \times 1 + 3 \times 3 \\ 4 \times 1 + 1 \times 3 \end{pmatrix} = \begin{pmatrix} 11 \\ 7 \end{pmatrix}$

$\therefore \ (1, \ 3) \ \rightarrow \ (11, \ 7)$

(ii) $\begin{pmatrix} 4 & -1 \\ 6 & 2 \end{pmatrix}\begin{pmatrix} 3 \\ 5 \end{pmatrix} = \begin{pmatrix} 4 \times 3 + (-1) \times 5 \\ 6 \times 3 + 2 \times 5 \end{pmatrix} = \begin{pmatrix} 7 \\ 28 \end{pmatrix}$

$\therefore \ (3, \ 5) \ \rightarrow \ (7, \ 28)$

(iii) $\begin{pmatrix} 7 & 4 \\ 5 & 3 \end{pmatrix}\begin{pmatrix} 2 \\ -3 \end{pmatrix} = \begin{pmatrix} 7 \times 2 + 4 \times (-3) \\ 5 \times 2 + 3 \times (-3) \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$\therefore \ (2, -3) \ \rightarrow \ (2. \ 1)$

(iv) $\begin{pmatrix} 4 & -1 \\ -8 & 5 \end{pmatrix}\begin{pmatrix} -3 \\ 5 \end{pmatrix} = \begin{pmatrix} 4 \times (-3) + (-1) \times 5 \\ (-8) \times (-3) + 5 \times 5 \end{pmatrix} = \begin{pmatrix} -17 \\ 49 \end{pmatrix}$

$\therefore \ (-3, \ 5) \ \rightarrow \ (-17, \ 49)$

2.4 Exercise

Question 1

Determine where each of the following points are moved to by the given matrix.

(i) $\begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix}\begin{pmatrix} 7 \\ 4 \end{pmatrix}$

$(7, 4) \rightarrow (\quad , \quad)$

(ii) $\begin{pmatrix} 4 & 3 \\ 5 & -2 \end{pmatrix}\begin{pmatrix} 6 \\ 1 \end{pmatrix}$

$(6, 1) \rightarrow (\quad , \quad)$

(iii) $\begin{pmatrix} -7 & 3 \\ 6 & -1 \end{pmatrix}\begin{pmatrix} 2 \\ -5 \end{pmatrix}$

$(2, -5) \rightarrow (\quad , \quad)$

(iv) $\begin{pmatrix} 5 & -4 \\ -6 & 2 \end{pmatrix}\begin{pmatrix} -3 \\ 1 \end{pmatrix}$

$(-3, 1) \rightarrow (\quad , \quad)$

Question 2
Given that,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x + 4y \\ 3x - y \end{pmatrix}$$

State the values of *a*, *b*, *c* and *d*

Question 3
Given that,

$$\begin{pmatrix} 5 & p \\ -6 & p \end{pmatrix}\begin{pmatrix} -3 \\ p \end{pmatrix} = \begin{pmatrix} 2p \\ -9p \end{pmatrix}$$

Find the value of *p*

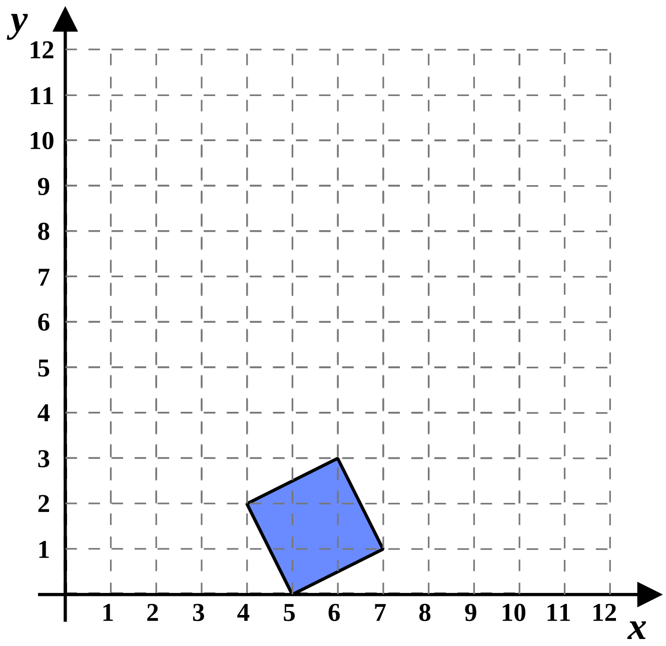
Question 4
Prove that any point on the line $y = x$ remains on the line $y = x$ when it is transformed by the matrix $\mathrm{M} = \begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix}$

HINT : Let a generalised point on $y = x$ be (*p*, *p*) and work out $\begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} p \\ p \end{pmatrix}$

Question 5

A square has vertices (5, 0), (7, 1), (6, 3) and (4, 2)

It is to be transformed by the matrix $\mathbf{M} = \begin{pmatrix} 2 & -3 \\ 1 & 2 \end{pmatrix}$



(i) Move each point by **M** and plot the resulting shape on the graph.

(ii) Work out the area of the original square.

(iii) Work out the area of the transformed shape.

(iv) There is a connecting between | **M** | and the areas of the two shapes.
Guess what this might be !