Lesson 8

Further A-Level Pure Mathematics
Complex Numbers : Core 1

8.1 When Circles Collide

Consider the following complex number equation with a view to plotting its locus on an Argand diagram,

$$|z + 2 - i| = |z - 4 + i|$$

From Lesson 7 it is clear that,

$$|z + 2 - i| = \lambda$$

or, equivalently, $|z - (-2 + i)| = \lambda$

describes a locus of a circle with centre $(-2,\ 1)$ and radius λ on an Argand diagram.

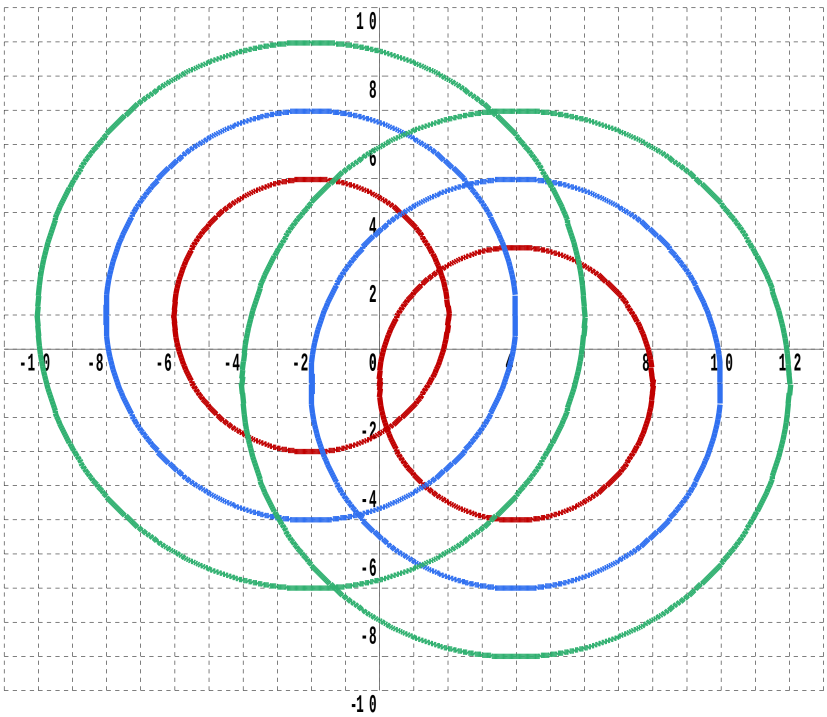
Similarly,

$$|z - 4 + i| = \lambda$$

or, equivalently, $|z - (4 - i)| = \lambda$

describes a locus of a circle with centre $(4,\ -1)$ and radius λ on an Argand diagram.

Equating the two can be thought of as generating a sequence of circles of the same radius, but with the radius increasing in steps, say $\lambda = 4, 6, 8, \ldots$



The red circles are of radius 4, the blue of radius 6 and the green of radius 8.
The points of interest are where red meets red, blue meets blue and green meets green.

The Key Question

When those points of intersection are all joined up what geometric shape is obtained ?

8.2 The Perpendicular Bisector

The quick answer to the key question is “a straight line”. However, from geometric work with a compass and straight edge, it can be seen that drawing a single pair of circles in this way is the method of obtaining a perpendicular bisector.
So, if one thinks about the piece of line between the circle centres of $(-2,\ 1)$ and $(4,\ -1)$, that locus of,

$$|z + 2 - i| = |z - 4 + i|$$

is the line that cuts it exactly in half and at right angles.

The equation of this perpendicular bisector could be approximately read off from the intersecting circles diagram.
Alternatively, the Year 1 material, tackled this very problem.

Year 1 Revision Question
Find the perpendicular bisector of the piece of line between the points
$(-2,\ 1)$ and $(4,\ -1)$

Teaching Video : http://www.NumberWonder.co.uk/Video/v9085(8).mp4

8.3 An Alternative Method

The equation of the perpendicular bisector described by,

$$|z + 2 - \mathrm{i}| = |z - 4 + \mathrm{i}|$$

can also be found by the following alternative method which uses the technique developed in Lesson 7.

$$|z + 2 - \mathrm{i}| = |z - 4 + \mathrm{i}|$$

$$|x + y\mathrm{i} + 2 - \mathrm{i}| = |x + y\mathrm{i} - 4 + \mathrm{i}|$$

$$|(x + 2) + (y - 1)\mathrm{i}| = |(x - 4) + (y + 1)\mathrm{i}|$$

Take the modulus of each side, and square to remove the square roots

$$(x + 2)^2 + (y - 1)^2 = (x - 4)^2 + (y + 1)^2$$

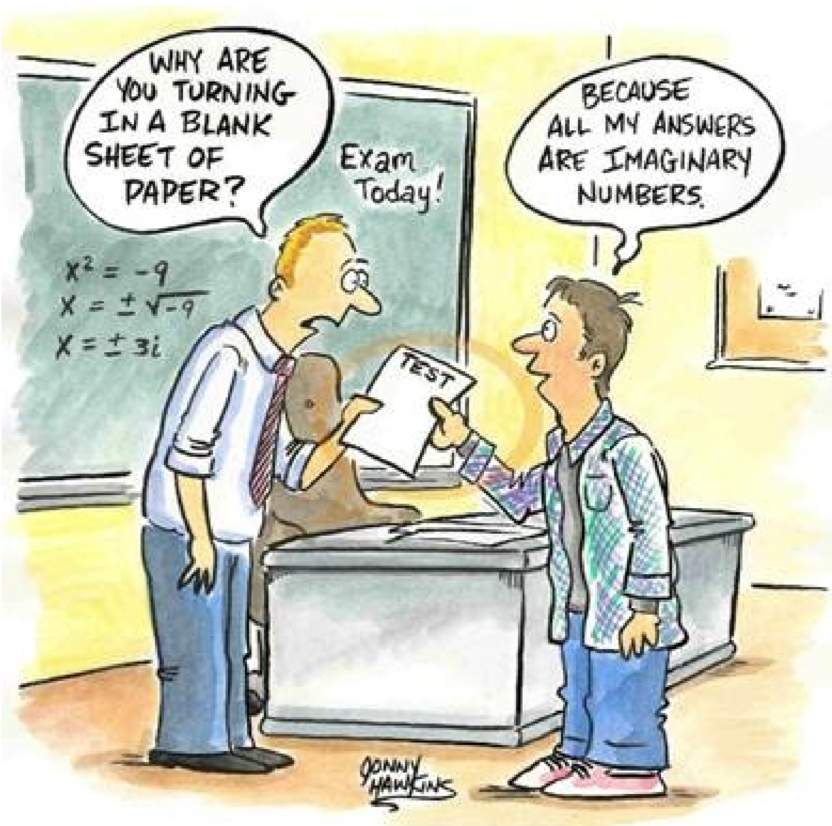
$$x^2 + 4x + 4 + y^2 - 2y + 1 = x^2 - 8x + 16 + y^2 + 2y + 1$$

$$4x - 2y + 5 = -8x + 2y + 17$$

$$12x - 12 = 4y$$

$$y = 3x - 3$$

This method will be required in the Exercise in Question 4 to get the locus of P and in Question 5 to get the requested equation of a circle.



8.4 Exercise

Question 1

(a) Write down the coordinates of the centre of the circle described by

$$|z - 10 - 5\mathrm{i}| = \lambda \qquad z \in \mathbb{C},\ \lambda \in \mathbb{R}$$

[1 mark]

(b) Write down the coordinates of the centre of the circle described by

$$|z + 6 - \mathrm{i}| = \lambda \qquad z \in \mathbb{C},\ \lambda \in \mathbb{R}$$

[1 mark]

(c) Hence, or otherwise, determine the Cartesian equation of the locus of z when

$$|z - 10 - 5\mathrm{i}| = |z + 6 - \mathrm{i}|$$

[3 marks]

(d) What will be the shortest distance between the origin and the locus found in part (c) ? In other words, what is the minimum value of $|z|$?

A sketch of the locus on an Argand diagram may help !

[3 marks]

Question 2

Further A-Level Examination Question from June 2010, FP2, Q6 (edited)

A complex number z is represented by the point P in the Argand diagram.

(a) Given that,

$$|z - 6| = |z|$$

sketch the locus of P

[2 marks]

Another complex number w is represented by the point Q in the Argand diagram.

(b) Given that,

$$|w - 3 - 4\text{i}| = 5$$

add a sketch of the locus of Q to your part (a) sketch

[2 marks]

(c) Find the complex numbers which satisfy both

$$|z - 6| = |z| \qquad \text{and} \qquad |w - 3 - 4\text{i}| = 5$$

[2 marks]

Question 3

Sketch on the same Argand diagram the locus of points satisfying,

(a) $|z - 2i| = |z - 8i|$

[2 marks]

(b) $|z - 6 - 2i| = 5$

[3 marks]

The complex number z satisfies both,

$$|z - 2i| = |z - 8i|$$

and $$|z - 6 - 2i| = 5$$

(c) Use your answers to parts (a) and (b) to find the two values of z

[2 marks]

Question 4

Further A-Level Examination Question from June 2014, F2, Q7

The point P represents a complex number z on an Argand diagram, where

$$|z + 1| = |2z - 1|$$

and the point Q represents a complex number w on the Argand diagram, where

$$|w| = |w - 1 + \mathrm{i}|$$

Find the exact coordinates of the points where the locus of P intersects the locus of Q

[7 marks]

Question 5

Further A-Level Exam Question from SAM, June 2017, FM Option 2, Paper 4, Q6

A curve has equation,

$$|z + 6| = 2|z - 6| \qquad z \in \mathbb{C}$$

(a) Show that the curve is a circle with equation

$$x^2 + y^2 - 20x + 36 = 0$$

[2 marks]

(b) Sketch the curve on an Argand diagram

[2 marks]