Lesson 7

Further A-Level Pure Mathematics
Complex Numbers : Core 1

7.1 Loci in the Argand Diagram

When faced with a piece of algebra such as

$$y = x^2 - 6x + 5$$

most mathematician's would immediately visualise this as a geometric object, a quadratic curve passing through (0, 5) on the *y*-axis. Even just this initial vague visualisation may be enough to answer a question.

Or perhaps more detail is needed.

Doing some algebra, factorising, leads to,

$$y = (x - 1)(x - 5)$$

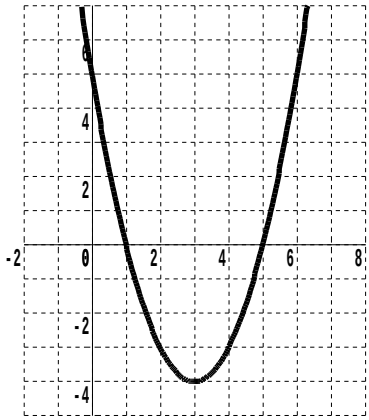
but again, faced with a piece of algebra, it is geometry that is in mind; the fact that this quadratic curve crosses the *x*-axis at (1, 0) and (5, 0).

Perhaps still more detail is required.

More algebra, this time completing the square, yields,

$$y = (x - 3)^2 - 4$$

and again, a visualisation that (3, – 4) is the minimum point is second nature.



$$y = x^2 - 6x + 5$$

With the foregoing in mind, it should not come as a surprise that, when faced with equations involving complex numbers, there are certain types of equation that are immediately visualised as a geometric object on an Argand diagram.
In general the geometric objects are often termed loci, and include familiar shapes such as straight lines, bits of straight lines, circles and ellipses.

7.2 The Circle

Given that

$$|z - 8 - 15i| = 6$$

(i) Derive the Cartesian equation of the locus of z

(ii) Sketch the locus of z on an Argand diagram

(iii) Calculate the minimum value of $|z|$

(iv) Find the maximum value, in radians, of $arg\ z$

Teaching Video : http://www.NumberWonder.co.uk/Video/v9085(7).mp4

7.3 Exercise

Question 1

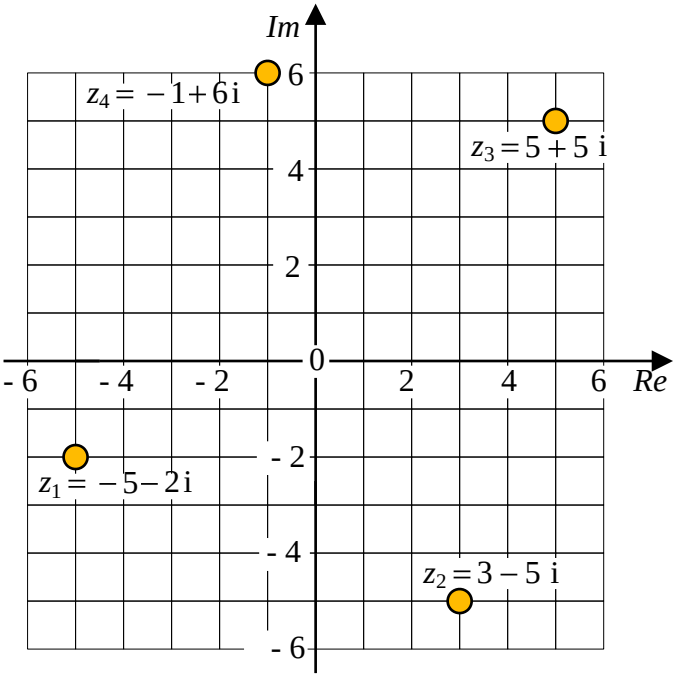
For each complex number on the Argand diagram find the principal argument. Note that your answers should be within the following ranges,

$$-\pi \leqslant arg(z_1) < -\frac{\pi}{2}$$

$$-\frac{\pi}{2} \leqslant arg(z_2) < 0$$

$$0 \leqslant arg\,(z_3) < \frac{\pi}{2}$$

$$\frac{\pi}{2} \leqslant arg(z_4) < \pi$$



Question 2

Further A-Level Examination Question from FP2 Mock Paper, Q8

A complex number z satisfies the equation,

$$|z - 5 - 12i| = 3$$

(a) Describe in geometrical terms with the aid of a sketch, the locus of the point which represents z in the Argand diagram.

[3 marks]

For the points on this locus, find

(b) the maximum and minimum values of $|z|$

[4 marks]

(c) the maximum and minimum values for $arg(z)$, giving your answers in radians to 2 decimal places.

[4 marks]

Question 3

Given that

$$|z - 24 - 7i| = 5$$

(i) Derive the Cartesian equation of the locus of z

(ii) Sketch the locus of z on an Argand diagram

(iii) Calculate the maximum value of $|z|$

(iv) Find the minimum value, in radians, of $arg(z)$

Question 4

The complex number z is defined by

$$z = \frac{3 + 5\mathrm{i}}{2 - \mathrm{i}}$$

(i) Find $|z|$

[4 marks]

(ii) *arg z*

[2 marks]

Question 5

The complex number z satisfies

$$|z + 3 - 6\mathrm{i}| = 3$$

Show that the exact maximum value of *arg z* in the interval $[-\pi, \pi]$ is

$$\frac{\pi}{2} + 2\,arcsin\left(\frac{1}{\sqrt{5}}\right)$$

[4 marks]

Question 6

$$z = -1 - \sqrt{3}\,\mathrm{i}$$

Find **(i)** $|z|$

[1 mark]

(ii) $\left|\frac{z}{z^*}\right|$

[4 marks]

(iii) $arg\, z$, $arg\,(z^*)$ and $arg\left(\frac{z}{z^*}\right)$

giving your answers in terms of π

[3 marks]

Question 7

The complex number w is given by

$$w = 6 + 3i$$

(i) Determine the value of $|w|$

[1 mark]

(ii) Find *arg w*, giving your answer in radians to 2 decimal places

[2 marks]

Given that

$$arg\,(\lambda + 5i + w) = \frac{\pi}{4}$$

where λ is a real constant

(iii) find the value of λ

[2 marks]