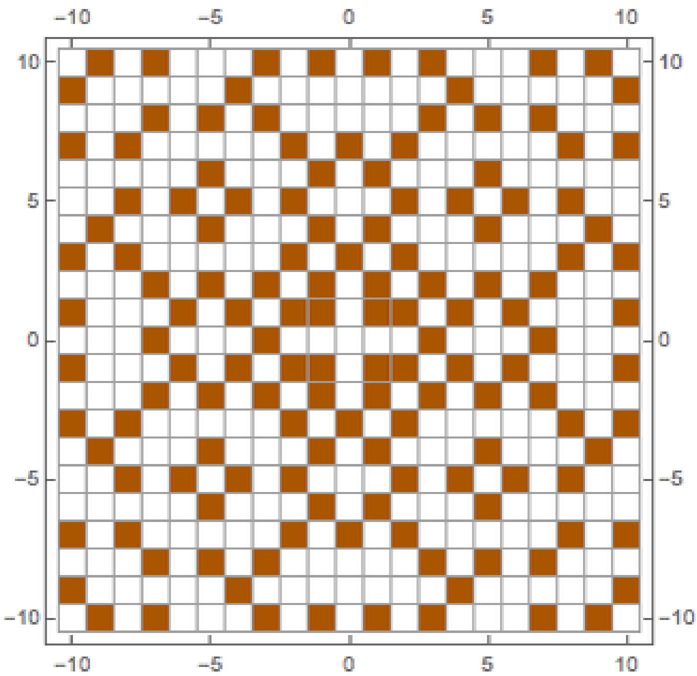
Lesson 6

Further A-Level Pure Mathematics
Complex Numbers : Core 1

6.1 The Gaussian Integers

The Gaussian integers are complex numbers of the form $a + b\text{i}$ where both a and b are integers. For this subset of complex numbers unique factorisation holds.
They are attractive to work with because each Gaussian integer is either a prime Gaussian integer or a composite Gaussian integer. A composite Gaussian integer can be decomposed into a unique product of prime Gaussian integers. So, some concepts, familiar from earlier work with the integers, get a new lease of life with the Gaussian integers. The statement, $z \in \mathbb{Z}[\text{i}]$ says "z is a Gaussian integer".



The integer grid shows Gaussian primes for values around zero.
On the real axis observe that 3 and 7 are Gaussian prime but 2 and 5 are not.

So what are the factors of these composite Gaussian integers ?
With a little thought and playing around,

2 factorises into $(1 + \text{i})\ (1 - \text{i})$ and 5 factorises into $(2 + \text{i})\ (2 - \text{i})$

From a careful look at the diagram, $(1 \pm \text{i})$ and $(2 \pm \text{i})$ are all Gaussian prime.
There is no further factorising to do and the unique prime factorisation of 2 and 5 in Gaussian integers has been found.
In Lesson 3, the number 6 was used to show that, in the general, complex numbers don't have unique factorisation because it factored in two ways,

$$6 = (1 + \text{i})\ (1 - \text{i}) \times 3$$

$$6 = \left(1 + \sqrt{5}\,\text{i}\right)\left(1 - \sqrt{5}\,\text{i}\right)$$

The second of these is not valid if only Gaussian integers are being considered.

6.2 The Square Root of a Complex Number

In the teaching video I will show you a very clever way to square root a Gaussian integer if it has an answer that is also a Gaussian integer.

Find z such that $z^2 - 5 - 12\text{i} = 0, \qquad z \in \mathbb{Z}[\text{i}]$

Teaching Video : http://www.NumberWonder.co.uk/Video/v9085(6).mp4

I am indebted to Dr Neha Agrawal of Murray Edwards College,
Cambridge University, for teaching me this method
of finding complex square roots in $\mathbb{Z}[\text{i}]$

6.3 Exercise

Question 1

A complex number $z = a + b\text{i}$, where a and b are real numbers, is sought that satisfies the equation,

$$z^2 - 7 - 24\text{i} = 0 \qquad z \in \mathbb{Z}[\text{i}]$$

(i) Show that $ab = 12$

(ii) Write down the value of $a^2 - b^2$

(iii) Hence find the two values of z such that

$$z^2 - 7 - 24\text{i} = 0 \qquad z \in \mathbb{Z}[\text{i}]$$

(iv) Show that your part (iii) solutions, when squared are equal to $7 + 24\text{i}$ as you would expect.

Question 2

A complex number $z = a + b\text{i}$, where a and b are real numbers, is sought that satisfies the equation,

$$z^2 + 9 - 40\text{i} = 0 \qquad z \in \mathbb{Z}[\text{i}]$$

(i) Show that $ab = 20$

(ii) Write down the value of $a^2 - b^2$

(iii) Hence find the two values of z such that

$$z^2 + 9 - 40\text{i} = 0 \qquad z \in \mathbb{Z}[\text{i}]$$

(iv) Show that your part (iii) solutions, when squared, are equal to $-9 + 40\text{i}$ as you would expect.

Question 3

A complex number $z = a + b\mathrm{i}$, where a and b are real numbers, is sought that satisfies the equation,

$$z^2 - 40 + 42\mathrm{i} = 0 \qquad z \in \mathbb{Z}[\mathrm{i}]$$

(i) Find the two possible values of z

(ii) The formula to solve quadratic equations of the form

$$a z^2 + bz + c = 0 \qquad z \in \mathbb{C}$$

where a, b and c are complex numbers is

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Show what happens when this formula is used to try and solve,

$$z^2 - 40 + 42\mathrm{i} = 0 \qquad z \in \mathbb{Z}[\mathrm{i}]$$

Question 4

(i) Work out $(1 + \sqrt{7}\,i)\,(1 - \sqrt{7}\,i)$

(ii) Work out $(1 + i)\,(1 - i)\,(1 + i)\,(1 - i)\,(1 + i)\,(1 - i)$

(iii) Explain how your part (i) and part (ii) answers show that complex numbers do not have unique factorisation.

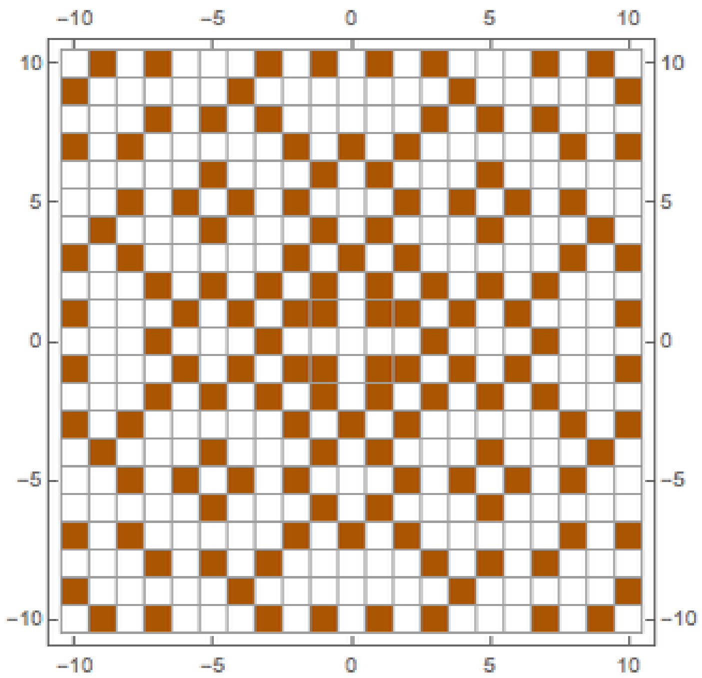
Question 5

A certain number p is a prime integer but not a Gaussian prime integer.
This is because when considered to be the Gaussian complex number it has the two factors,

$$(3 + 2i) \quad \text{and} \quad (3 - 2i)$$

What number is p ?

Question 6



(i) On the grid of Gaussian integers, circle $6 + 5i$

(ii) Hence state if $6 + 5i$ is a Gaussian prime or not.

Question 7

A polynomial $p(z)$ has some coefficients that are complex numbers,

$$z^3 + \mathrm{i}\, z^2 + 6z = 0 \qquad z \in \mathbb{C}$$

The formula to solve quadratic equations of the form

$$a\, z^2 + bz + c = 0 \qquad z \in \mathbb{C}$$

where a, b and c are complex numbers is

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2\, a}$$

Show how this formula may be used to help find the roots of $p(z)$

Question 8

A complex number $z = a + b\text{i}$, where a and b are real numbers, is sought that satisfies the equation,

$$z^2 + 3 - 2\text{i} = 0 \qquad z \in \mathbb{Z}[\text{i}]$$

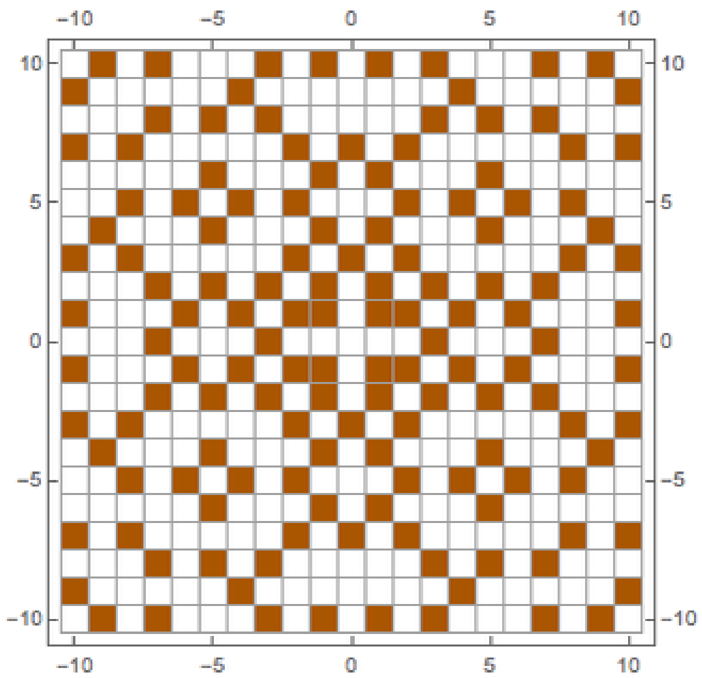
(i) Show that $ab = 1$

(ii) Find the value of $a^2 - b^2$

(iii) Explain why there is no Gaussian integer that satisfies the equation

$$z^2 + 3 - 2\text{i} = 0 \qquad z \in \mathbb{Z}[\text{i}]$$

(iv) On the grid of Gaussian integers, circle the complex number $-3 + 2\text{i}$



(v) Explain how your part (iv) answer confirms that there is no complex number that satisfies the equation

$$z^2 + 3 - 2\text{i} = 0 \qquad z \in \mathbb{Z}[\text{i}]$$

Question 9

A-Level Examination Question from June 2004, P4, Q3

The complex number $z = a + b\mathrm{i}$, where a and b are real numbers, satisfies the equation,

$$z^2 + 16 - 30\mathrm{i} = 0$$

(a) Show that $ab = 15$

[2 marks]

(b) Write down a second equation in a and b and hence find the roots of

$$z^2 + 16 - 30\mathrm{i} = 0$$

[4 marks]