Lesson 5

Further A-Level Pure Mathematics
Complex Numbers : Core 1

5.1 Modulus

For a given a complex number, $z = a + b\text{i}$, the modulus is denoted $|z|$.
It is the distance from the origin to that number on an Argand diagram.
In other words,

$$|z| = \sqrt{a^2 + b^2}$$

5.2 The Principal Argument

For a given complex number, $z = a + b\text{i}$, the principal argument is denoted $Arg(z)$.
It is the angle between the positive real axis and the line joining the point of interest to the origin on an Argand diagram. The calculation varies depending upon which of the four quadrants on the Argand diagram the point resides. This will be looked at shortly in this Lesson's teaching video.

First, however, some scene setting.

As always in mathematics an anticlockwise rotation is considered to be positive.
and a clockwise rotation negative.
By convention, arguments are given in radians and, to make the function one-to-one, in the range,

$$-\pi < arg(z) \leqslant \pi$$

Thus, to rotate from the positive x-axis direction onto points in either the 1st or the 2nd quadrants the rotation is given as a positive (anticlockwise) rotation.

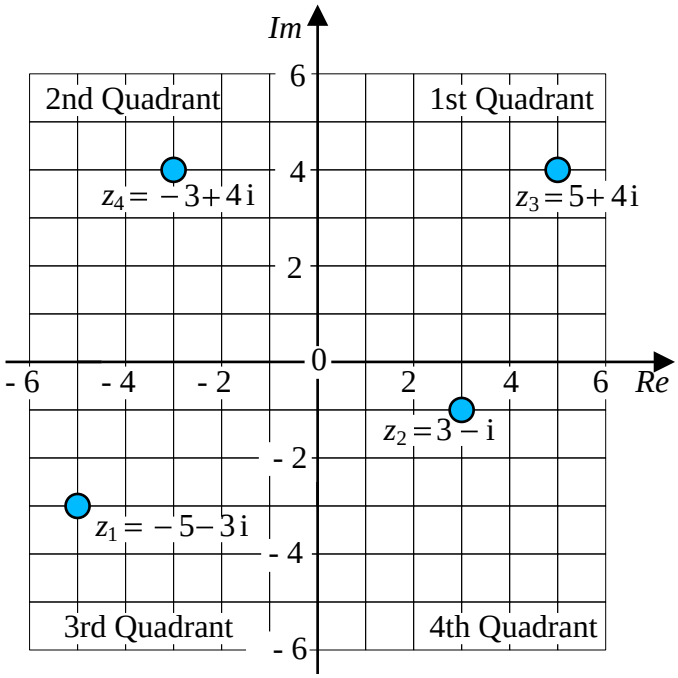
To rotate from the positive x-axis direction onto points in either the 3rd or the 4th quadrants the rotation is given as a negative (clockwise) rotation.

An argument given in this range is formally termed a “Principal Argument” but often the word “principal” is dropped. In examination questions on arguments, answers should always be given in the range $-\pi < arg(z) < \pi$ unless otherwise instructed.

An argument outside the range $-\pi < arg(z) < \pi$ can be brought back into range by simply adding or subtracting a multiple of 2π. However, it is better to get into the habit of generating answers within the required range directly rather than having to “rescue” out of range answers.

In University courses an argument outside of the range $-\pi < arg(z) < \pi$ is written $arg\,(z)$ whereas a principle argument is denoted $Arg\,(z)$, with the capital letter A.
In the Further Mathematics A-level course this distinction is not made, although teachers often, without thinking, (correctly) put the capital letter in !

5.3 Example on Finding Principal Arguments



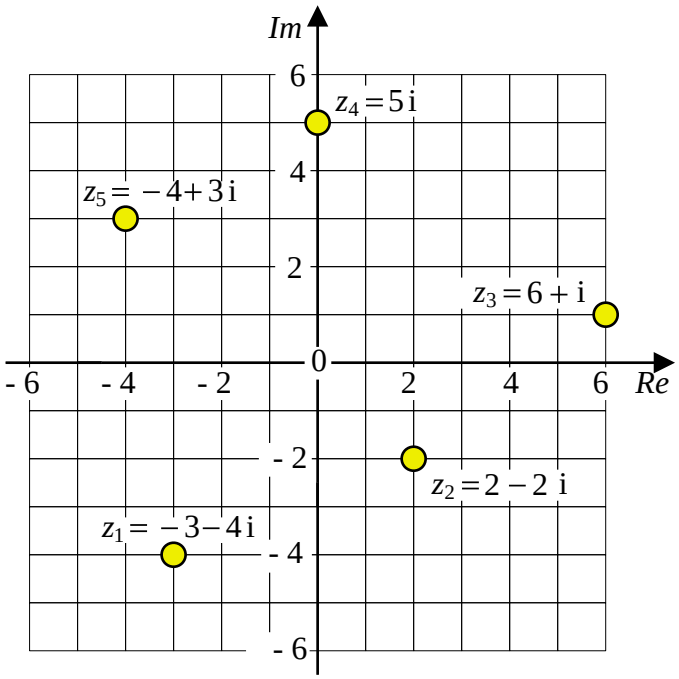
For each of the complex numbers on the Argand diagram find the principal argument.

Teaching Video : http://www.NumberWonder.co.uk/Video/v9085(5).mp4

5.5 Exercise

Question 1

For each of the complex numbers on the Argand diagram find the principal argument.



Question 2

Further A-Level Examination Question from January 2012, FP1, Q1

Given that

$$z_1 = 1 - i$$

(a) find $arg(z_1)$

[2 marks]

Given also that $z_2 = 3 + 4i$, find, in the form $a + ib$, $a, b \in \mathbb{R}$

(b) $z_1 z_2$

[2 marks]

(c) $\frac{z_2}{z_1}$

[3 marks]

In part (b) and part (c) you must show all your working clearly.

Question 3

$$z = \frac{26}{2 - 3\mathrm{i}}$$

(a) Find z in the form $a + b\mathrm{i}$ where $a, b \in \mathbb{R}$

[2 marks]

(b) Find z^2 in the form $a + b\mathrm{i}$ where $a, b \in \mathbb{R}$

[2 marks]

(c) Determine $|z|$

[2 marks]

(d) Calculate $arg(z^2)$ giving your answer in radians to 2 decimal places

[2 marks]

Question 4

Further A-Level Examination Question from January 2007, FP1, Q3

The complex numbers z_1 and z_2 are given by

$$z_1 = 5 + 3\text{i}$$

$$z_2 = 1 + p\text{i}$$

where p is an integer.

(a) Find $\frac{z_2}{z_1}$ in the form $a + \text{i}b$, where a and b are expressed in terms of p

[3 marks]

Given that,

$$arg\left(\frac{z_2}{z_1}\right) = \frac{\pi}{4}$$

(b) find the value of p

[2 marks]

Question 5

Further A-Level Examination Question from January 2014, IAL, FP1, Q5

$$z = 5 + i\sqrt{3} \qquad\qquad w = \sqrt{3} - i$$

(a) Find the value of $|w|$

[1 mark]

Find in the form $a + ib$, where a and b are real numbers,

(b) zw, showing clearly how you obtained your answer

[2 marks]

(c) $\frac{z}{w}$, showing clearly how you obtained your answer

[3 marks]

Given that

$$arg(z + \lambda) = \frac{\pi}{3}, \qquad \text{where } \lambda \text{ is a real constant}$$

(d) find the value of λ

[2 marks]

Question 6

Further A-Level Examination Question from January 2019, IAL, F1, Q9

The complex numbers z_1 and z_2 are given by

$$z_1 = -1 - \mathrm{i} \qquad \text{and} \qquad z_2 = 3 - 4\mathrm{i}$$

(a) Find the argument of the complex number $z_1 - z_2$

Give your answer in radians to 3 decimal places.

[3 marks]

(b) Find the complex number $\frac{z_1}{z_2}$ in the form $a + b\mathrm{i}$

where a and b are rational numbers.

[3 marks]

(c) Find the modulus of $\frac{z_1}{z_2}$ giving your answer as a simplified surd.

[2 marks]

(d) Find the values of the real constants p and q such that

$$\frac{p + \mathrm{i}q - 8z_1}{p - \mathrm{i}q - 8z_2} = 3\mathrm{i}$$

[5 marks]