3.1 Roots and the Complex Plane

Lesson 1, Question 7 asked for a proof that $z = 1 + i$ is a solution to

$$z^4 + 4 = 0$$

It then asked that another solution to this quartic equation be guessed.

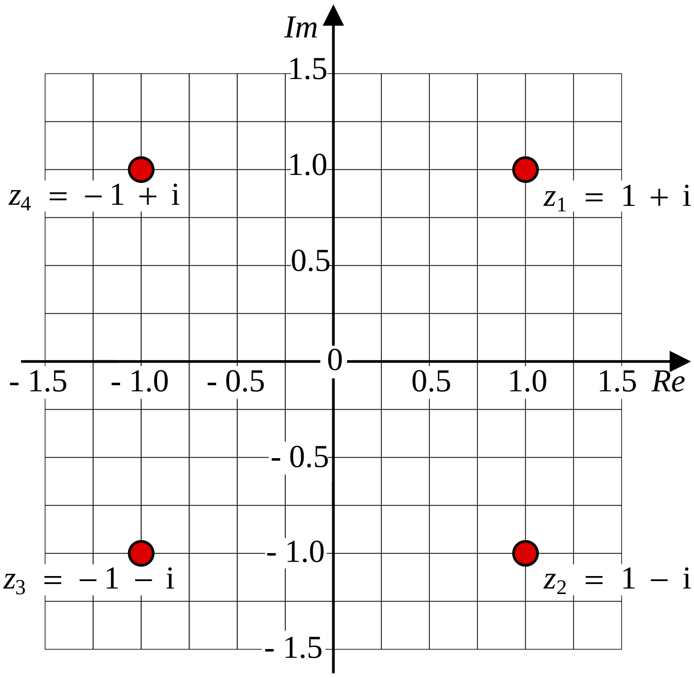
In fact this quartic equation (a polynomial of degree 4) has the four complex roots,

$z_1 = 1 + i \qquad z_2 = 1 - i \qquad z_3 = -1 - i \qquad z_4 = -1 + i$

A two dimensional picture, called an Argand Diagram, can be drawn to show where the roots of this quartic equation lie on the complex plane.
Given the root of a polynomial, $z_{ROOT} = a + b$i the real part of the number a is plotted on the x-axis and the imaginary part of the number, b, is plotted on the y-axis.
In fact, mathematician's stop using the terms x-axis and y-axis and talk instead of the *Real*-axis, and the *Imaginary*-axis and the two dimensional surface in which the real and imaginary axes reside is termed the complex plane.

An Argand Diagram of the Complex Plane
showing the four complex roots of the equation

$$z^4 + 4 = 0$$



One method of verifying that each of these is a roots is valid would be to substitute them, one at a time into the quartic and show that they make it true, as in Lesson 1.

3.2 The Fundamental Theorem of Algebra

The Fundamental Theorem of Algebra states that any polynomial in z of degree n has exactly n roots, where some or all of the roots may be complex numbers.
The polynomial can even have complex coefficients.

This theorem tells us that, $z^4 + 4 = 0$, which is a polynomial in z of degree 4, has four roots. So once four roots have been found, there is no need to look for any more.

3.3 Unique Factorisation

The Fundamental Theorem of Algebra ties in with fact that for any polynomial in z of degree one or more unique factorisation holds.

When working with the integers, Z, The Fundamental Theorem of Arithmetic tells us that every number that in not prime can be written as a product of primes in, essentially, only one way.
For example,

$$120 = 2^3 \times 3 \times 5$$

The "only one way" is formally referred to as "unique factorisation".

Previously, polynomials in x with integer coefficients have been studied and these also have unique factorisation.
For example,

$$x^3 + 4x^2 + 4x + 3 = (x^2 + x + 1)(x + 3)$$

The factor theorem is of use in trying to find the unique factorisation of such polynomials. In fact, the unique factorisation holds for real number coefficients.

With polynomials in z there is the restriction that the degree of the polynomial must be one or more. To understand the restriction consider the polynomial in z of degree 0,

$$p(z) = 6z^0$$

This does not have a unique factorisation.

On the one hand,

$$6 = (1 + \mathrm{i})\ (1 - \mathrm{i}) \times 3$$

and on the other,

$$6 = \left(1 + \sqrt{5}\,\mathrm{i}\right)\left(1 - \sqrt{5}\,\mathrm{i}\right)$$

This single example, a counter example, is enough to show that unique factorisation does not hold for complex polynomials in z of degree 0

3.4 Solving Polynomial Equations

Complex Conjugate Pair
If $p(z)$ is a polynomial with real coefficients and z_1 is a complex root of $p(z) = 0$ then z_1^* is also a root of $p(z) = 0$.
Such roots are termed a complex conjugate pair.

Further A-Level Examination Question from January 2009, FP1, Q2

$$f(z) = 2z^4 - 14z^3 + 33z^2 - 26z + 10$$

Given $z = 3 + \mathrm{i}$ is a solution of the equation $f(z) = 0$, solve $f(z) = 0$ completely.

Teaching Video : http://www.NumberWonder.co.uk/Video/v9085(3).mp4

3.5 Exercise

Question 1

Given that $z = 3$ is the real solution of the equation,

$$z^3 - 5z^2 + 11z - 15 = 0$$

find the two complex solutions of this equation.

Question 2

$$f(z) = z^3 - 6z^2 + 21z - 26$$

(i) Show that $f(2) = 0$

[1 mark]

(ii) Hence solve $f(z) = 0$ completely.

[3 marks]

Question 3

Further A-Level Examination Question from June 2014, FR1 (R), Q1

The roots of the equation,

$$2z^3 - 3z^2 + 8z + 5 = 0$$

are z_1, z_2 and z_3.

Given that $z_1 = 1 + 2\text{i}$, find z_2 and z_3.

[5 marks]

Question 4

Further A-Level Examination Question from June 2009, FP1, Q1

Given that $x = \mathrm{i}$ is a complex solution of the equation,

$$x^4 + 6x^3 + 26x^2 + 6x + 25 = 0$$

(a) express $x^4 + 6x^3 + 26x^2 + 6x + 25$ as a product of two quadratic factors

[4 marks]

(b) Hence solve completely the equation $x^4 + 6x^3 + 26x^2 + 6x + 25 = 0$

[2 marks]

Question 5

Further A-Level Examination Question from June 2018, F1, Q5

Given that

$$z^4 - 6z^3 + 34z^2 - 54z + 225 = (z^2 + 9)(z^2 + az + b)$$

where *a* and *b* are real numbers,

(a) find the value of *a* and the value of *b*

[2 marks]

(b) Hence find the exact roots of the equation

$$z^4 - 6z^3 + 34z^2 - 54z + 225 = 0$$

[4 marks]

(c) Show your roots on a single Argand diagram

[2 marks]

Question 6

Further A-Level Examination Question from June 2013, FP1, Q3

Given that $x = \frac{1}{2}$ is a root of the equation,

$$2x^3 - 9x^2 + kx - 13 = 0 \qquad k \in \mathbb{Z}$$

find

(a) the value of k

[3 marks]

(b) the other two roots of the equation

[4 marks]

Question 7

Further A-Level Examination Question from January 2010, FP1, Q6

Given that 2 and 5 + 2i are roots of the equation

$$z^3 - 12z^2 + cz + d = 0, \qquad c, d \in \mathbb{R}$$

(a) write down the other complex root of the equation

[1 mark]

(b) Find the value of c and the value of d

[5 marks]

(c) Show the three roots of this equation on a single Argand diagram.

[2 marks]

Question 8

Further A-Level Examination Question from June 2006, FP1, Q4

Given that 3 - 2i is a solution of the equation

$$x^4 - 6x^3 + 19x^2 - 36x + 78 = 0$$

(a) solve the equation completely

[7 marks]

(b) Show on a single Argand diagram the four points that represent the roots of the equation.

[2 marks]

Question 9

Find the four roots of the equation

$$z^4 - 16 = 0$$