Lesson 2

Further A-Level Pure Mathematics
Complex Numbers : Core 1

2.1 Complex Number Division

A complex number, z, is a number of the form,

$$z = a + b\mathrm{i}$$

where a and b are real numbers, $\mathrm{i} = \sqrt{-1}$ and $\mathrm{i}^2 = -1$

Given two complex numbers,

$$z = a + b\mathrm{i} \qquad \text{and} \qquad w = c + d\mathrm{i}$$

the arithmetic operations of addition, subtraction and multiplication can be performed with the two numbers and the result is in the form of a complex number.

Added :

$$\begin{aligned} z + w &= (a + b\mathrm{i}) + (c + d\mathrm{i}) \\ &= (a + c) + (b + d)\mathrm{i} \end{aligned}$$

As a, b, c and d are all real numbers, $(a + c)$ and $(b + d)$ will also be real numbers which we could call u and v.

$\therefore\ z + w = u + v\mathrm{i}$ which is in the form of a complex number

Subtracted :

$$\begin{aligned} z - w &= (a + b\mathrm{i}) - (c + d\mathrm{i}) \\ &= (a - c) + (b - d)\mathrm{i} \end{aligned}$$

As a, b, c and d are all real numbers, $(a - c)$ and $(b - d)$ will also be real numbers which we could call u and v.

$\therefore\ z - w = u + v\mathrm{i}$ which is in the form of a complex number

Multiplied :

$$\begin{aligned} zw &= (a + b\mathrm{i})\ (c + d\mathrm{i}) \\ &= ac + ad\mathrm{i} + bc\mathrm{i} + bd\,\mathrm{i}^2 \\ &= ac + ad\mathrm{i} + bc\mathrm{i} + bd\ (-1) \\ &= (ac - bd) + (ad + bc)\mathrm{i} \end{aligned}$$

As a, b, c and d are all real numbers, $(ac - bd)$ and $(ad + bc)$ will also be real numbers which we could call u and v.

$\therefore\ zw = u + v\mathrm{i}$ which is in the form of a complex number

This all leads to an obvious question; “What about division ?”.

Can any complex number, divided by any other (non zero) complex number be manipulated into the form of a complex number ?

Teaching Video : http://www.NumberWonder.co.uk/Video/v9085(2).mp4

Division : $\frac{z}{w} = \frac{a + b\mathrm{i}}{c + d\mathrm{i}}$

2.2 Exercise

Question 1

Three complex numbers are,

$$u = 5 - 3\mathrm{i} \qquad v = 4 + 2\mathrm{i} \quad \text{and} \quad w = \frac{u}{v}$$

(i) Write down v^* the complex conjugate of v.

(ii) Show how to use v^* to write w in the form $a + b\mathrm{i}$

Question 2

Given the complex number,

$$u = 3 + \mathrm{i}$$

write $\dfrac{u}{u^*}$ in the form $a + b\mathrm{i}$

Question 3

Given that,

$$z_1 = 1 + i \qquad z_2 = 2 + i \quad \text{and} \quad z_3 = 3 + i$$

write each of the following in the form $a + bi$,

(i) $\dfrac{z_1 z_2}{z_3}$

(ii) $\dfrac{(z_2)^2}{z_1}$

(iii) $\dfrac{2z_1 + 5z_3}{z_2}$

Question 4

Simplify

$$\frac{2 + i}{2 - i} - \frac{2 - i}{2 + i}$$

Question 5

Given that

$$w = \sqrt{3} + \sqrt{2}\,i$$

write w^{-1} in the form $a + bi$

Question 6

Simplify,

$$\frac{1}{i} + \frac{1}{1 + i} + \frac{1}{1 - i}$$

Question 7

By using the binomial expansion, or otherwise, show that,

$$(1 + 2i)^5 = 41 - 38i$$

Question 8

The complex number z is defined by,

$$z = \frac{3 + q\mathrm{i}}{q - 5\mathrm{i}} \qquad \text{where } q \in \mathbb{R}$$

Given that the real part of z is $\frac{1}{13}$,

(i) find the possible values of q

(ii) write the possible values of z in the form $a + b\mathrm{i}$ where a and b are real constants.

Question 9

$$z = 4 - \mathrm{i}\sqrt{2}$$

Use algebra to express $\dfrac{z+4}{z-3}$ in the form $p + q\mathrm{i}\sqrt{2}$

where p and q are rational numbers.