Lesson 6

A-Level Pure Mathematics
Sequences & Series : Year 2

6.1 Primes &Arithmetic Progressions

Some Arithmetic Progressions contain no primes,

$$AP\{4,10\} \;:\; 4,\; 14,\; 24,\; 34,\; 44,\; 54,\; 64,\; 74,\; 84,\; 94,\; \ldots$$

Some contain only an initial term that is prime,

$$AP\{5,10\} \;:\; \mathbf{5},\; 15,\; 25,\; 35,\; 45,\; 55,\; 65,\; 75,\; 85,\; 95,\; \ldots$$

Some contain many primes,

$$AP\{9,10\} \;:\; 9,\; \mathbf{19},\; \mathbf{29},\; 39,\; 49,\; \mathbf{59},\; 69,\; \mathbf{79},\; \mathbf{89},\; 99,\; \ldots$$

These three examples are amongst those depicted below,

$y = 0 + 10n$										
$y = 1 + 10n$		11		31	41		61	71		
$y = 2 + 10n$	2									
$y = 3 + 10n$	3	13	23		43	53		73	83	
$y = 4 + 10n$										
$y = 5 + 10n$	5									
$y = 6 + 10n$										
$y = 7 + 10n$	7	17		37	47		67			97
$y = 8 + 10n$										
$y = 9 + 10n$		19	29			59		79	89	

6.2 Proof

Theorem : No number in the arithmetic progression $AP\{4,10\}$ is prime.
Proof : All terms in $AP\{4,10\}$ are of the form.

$$y = 4 + 10n, \qquad for \;\; n = 0,\; 1,\; 2,\; 3,\; \ldots$$

$$y = 2\,(\,2 + 5n\,)$$

This shows that these numbers are always divisible by 2.
The first term, 4, is not prime, and so no term in $AP\{4,10\}$ is prime □

6.3 Trapping the Primes

The “ten wide”number grid shows that, with the exception of 2 and 5, all primes must be in one of $AP\{1,10\}$, $AP\{3,10\}$, $AP\{7,10\}$ or $AP\{9,10\}$.
When a prime except 2 or 5 is divided by 10 the remainder must be one of 1, 3, 7 or 9.
Also, when an integer is divided by 10 if the remainder is 0, 2, 4, 5, 6 or 8 then that number cannot be prime.

6.4 Exercise

Question 1

Prove the following theorem,

Theorem : No number in the arithmetic progression $AP\{6,10\}$ is prime.

Question 2

George says that he has a number that when divided by 10 has remainder 7 and so the number must be prime.

Give an example that proves George is wrong.

Question 3

Write down a prime number that is in $AP\{5,10\}$.

Question 4

Disprove the following theorem,

Theorem : No number in the arithmetic progression $AP\{2,10\}$ is prime.

Question 5

The “ten wide” number grid trapped the primes in four rows, with the exception of the prime numbers 2 and 5. Is “ten wide” the best “trapper of primes” ?

A “six wide” trapper is shown below.

$y = 0+6n$																	
$y = 1+6n$		7	13	19		31	37	43			61	67	73	79			97
$y = 2+6n$	2																
$y = 3+6n$	3																
$y = 4+6n$																	
$y = 5+6n$	5	11	17	23	29		41	47	53	59		71		83	89		101

With the exception of 2 and 3, in which two Arithmetic Progressions does this number grid suggest all of the primes may lie ?

Question 6
Prove the following theorem,
Theorem : No number in the arithmetic progression $AP\{4,6\}$ is prime.

Question 7
When a prime number, other than 2 or 3, is divided by 6 what must the remainder be one of ?

Question 8
A number, x, when divided by 6 has remainder 2.
What can be said about the nature of x ?

Question 9
Use the number grid below to produce a “seven wide” prime trapper.

$y = 0+7n$														
$y = 1+7n$														
$y = 2+7n$														
$y = 3+7n$														
$y = 4+7n$														
$y = 5+7n$														
$y = 6+7n$														

Is this a good prime trapper ?
Explain your answer.

Question 10
In 1837 Peter Dirichlet [1805-1859] proved that for any two positive coprime integers, a and d, there are infinitely many primes of the form $y = d + an$
Of the numbers 0, 1, 2, 3, 4, 5, 6, and 7, which are coprime to 8 ?

Question 11
Which *AP*s in an "eight wide" number grid will contain infinitely many primes ?

Question 12
Which *AP*s in an "nine wide" number grid will contain infinitely many primes ?

Question 13
Consider arithmetic progressions of the form $AP\{a,16\}$

(i) For what values of a will the sequence contain infinitely many primes ?

(ii) List 3 composite numbers in $AP\{3,16\}$

(iii) List three prime numbers in $AP\{3,16\}$

Question 14
For any given positive integer n, greater than 1, the number of integers coprime to n is given a special name, the Euler totient function, φ.
Thus for an "n wide" number grid the number of as for which $AP\{a,n\}$ contains an infinite number of primes is given by $\phi(n)$.
To work out $\phi(n)$ use the fact that,

$$\phi\left(p^m\right) = p^{m-1}(p-1) \qquad \textit{where } p \textit{ is prime}$$

For the "seven wide" number grid,

$$\begin{aligned}\phi\left(7^1\right) &= 7^{1-1}(7-1)\\ &= 7^0 \times 6\\ &= 6\end{aligned}$$

Thus the primes are trapped in six of the seven rows of the number grid.

For the "ten wide" number grid, we utilise the multiplicative property the ϕ-function.

$$\begin{aligned}\phi(10) &= \phi(2 \times 5)\\ &= \phi\left(2^1\right) \times \phi\left(5^1\right)\\ &= 2^0 \times 1 \times 5^0 \times 4\\ &= 4\end{aligned}$$

Thus the primes are trapped in four of the ten rows of the number grid.

Use Euler's totient function, ϕ, to work out the number of rows the primes are trapped in when the number grid has the following widths.
Use the facts that,

$$\phi(p^m) = p^{m-1}(p-1) \qquad \textit{where } p \textit{ is prime}$$

$$\phi(p^m q^n) = \phi(p^m) \times \phi(q^n) \qquad \textit{where } p \textit{ and } q \textit{ are coprime}$$

(i) 11

(ii) 6

(iii) 8

(iv) 30

Question 15

Now that you know about Euler's totient function, ϕ, you may like to use it to think about and investigate which widths of number grid are best at trapping the primes into relatively tight spaces. For each width what percentage of the available number of rows are the primes restricted to ?