Lesson 7

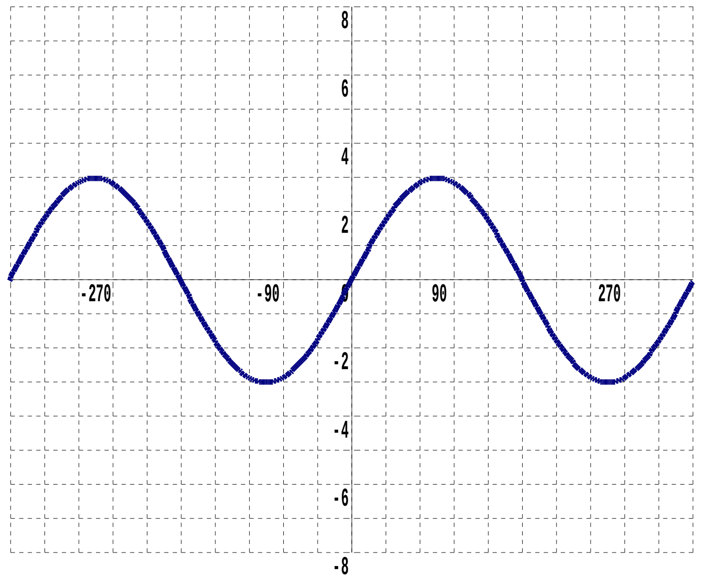
A-Level Pure Mathematics : Year 2
Trigonometric Identities

7.1 Addition of Trigonometric Waveforms

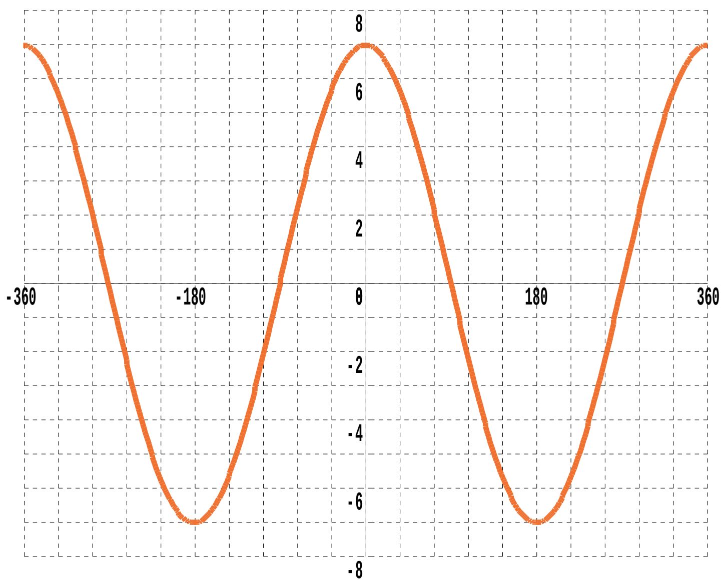
A surprising omission in the trigonometric expressions considered thus far are simple sums of *sine* and *cosine* such as, for example,

$$y = 3\sin\theta + 7\cos\theta$$

What makes this tricky to get a grip of is that there are no squares of trigonometric functions to manipulate. As a fallback strategy, graphs can be considered.
The obvious question to ask is "how complicated is the resulting waveform" ?

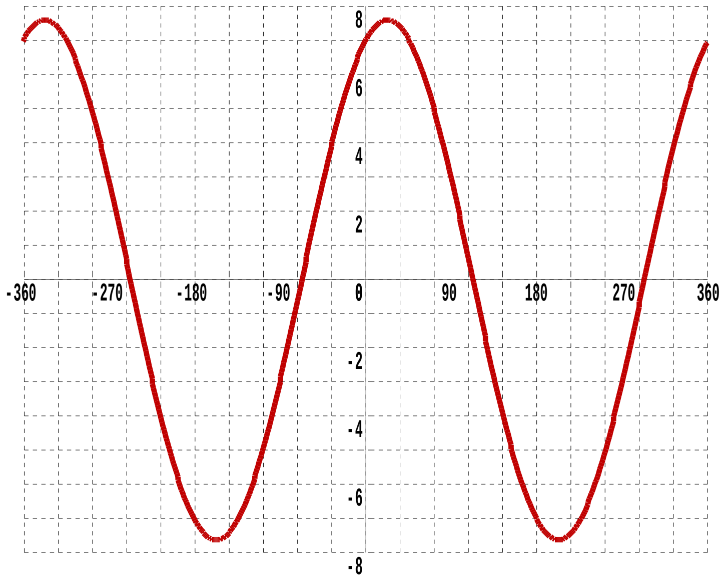


$$y = 3\sin\theta$$



$$y = 7\cos\theta$$

The resulting waveform for $y = 3\sin\theta + 7\cos\theta$ is surprisingly simple !



The waveform for $y = 3\,sin\,\theta + 7\,cos\,\theta$ is not complicated at all !
It's a *sine* wave moved left about 65° and height between 7 and 8.

Knowing that the resulting wave is a *sine* wave is the key to getting an exact answer because the form of the answer has to be,

$$R\,sin\,(\theta + \alpha)$$

where α is the shift left, and R is the height or *amplitude*.

Question : Express $3\,sin\,\theta + 7\,cos\,\theta$ in the form $R\,sin(\theta + \alpha)$ for $0 < \alpha < 90°$

Answer:

$$R\,sin(\theta + \alpha) = R\,sin\,\theta\,cos\,\alpha + R\,cos\,\theta\,sin\,\alpha$$

which is required to be $3\,sin\,\theta + 7\,cos\,\theta$

$$\therefore R\,cos\,\alpha = 3 \quad \text{and} \quad R\,sin\,\alpha = 7$$

Solving these two equations simultaneously by division,

$$\frac{R\,sin\,\alpha}{R\,cos\,\alpha} = \frac{7}{3}$$

$$\alpha = arctan\left(\frac{7}{3}\right)$$

$$\alpha = 66.8°$$

For reasons to be explained shortly,

$$R = \sqrt{7^2 + 3^2}$$

$$= 7.62$$

Conclusion :

$$3\,sin\,\theta + 7\,cos\,\theta = 7.62\,sin(\theta + 66.8°)$$

7.2 Why applying Pythagoras' theorem gives the value of *R*

$$\sqrt{(R\sin\alpha)^2 + (R\cos\alpha)^2} = \sqrt{R^2\sin^2\alpha + R^2\cos^2\alpha}$$

$$= \sqrt{R^2(\sin^2\alpha + \cos^2\alpha)}$$

$$= \sqrt{R^2} \qquad \text{because} \quad \cos^2\alpha + \sin^2\alpha = 1$$

$$= R$$

7.3 Example

Write $8\sin\theta + 15\cos\theta$ in the form $R\sin(\theta + \alpha)$ for $0 < \alpha < 90°$

Teaching Video : http://www.NumberWonder.co.uk/v9040/7.mp4

After watching the Teaching Video, write out a solution in the space below,

☞

[4 marks]

7.4 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 40

Question 1

(i) Write $2\sin\theta + \sqrt{5}\cos\theta$ in the form $R\sin(\theta + \alpha)$ for $0 < \alpha < 90°$

[4 marks]

(ii) Keeping in mind that the maximum that $\sin(\theta + \alpha)$ can be, regardless of α, is 1, what is the maximum value of $2\sin\theta + \sqrt{5}\cos\theta$?

[1 mark]

(iii) Keeping in mind that the minimum that $\sin(\theta + \alpha)$ can be, regardless of α, is – 1, what is the minimum value of $2\sin\theta + \sqrt{5}\cos\theta$?

[1 mark]

(iv) Solve the equation $2\sin\theta + \sqrt{5}\cos\theta = 1.5$
Give both solutions that are in the interval $0° < \theta < 360°$

[4 marks]

Question 2

(i) Write $\sqrt{3}\ sin\,\theta\ +\ cos\,\theta$ in the form $R\,sin\,(\ \theta\ +\ \alpha\ )$ for $0\ <\ \alpha\ <\ 90°$

[4 marks]

(ii) What is the minimum value of $\sqrt{3}\ sin\,\theta\ +\ cos\,\theta$?

[1 mark]

(iii) What is the maximum value of $3\,sin\,\theta\ +\ \sqrt{3}\ cos\,\theta$?

[1 mark]

(iv) Solve the equation $\sqrt{3}\ sin\,\theta\ +\ cos\,\theta\ =\ \sqrt{2}$

Give both solutions that are in the interval $0° < \theta < 360°$

[4 marks]

Question 3

(i) Write $\sin\theta + \cos\theta$ in the form $R\sin(\theta + \alpha)$ for $0 < \alpha < 90°$

[3 marks]

(ii) What is the minimum value of $\sin\theta + \cos\theta$?

[1 mark]

(iii) What is the exact maximum value of $\dfrac{1}{\sin\theta + \cos\theta + 3}$?

[2 mark]

(iv) Solve the equation $\sin\theta + \cos\theta = 1$ over the interval $0 \leqslant \theta \leqslant 360°$ giving all solutions as exact values.

[4 marks]

Question 4

Find a formula for α in terms of A and B when $A \sin\theta + B \cos\theta$ is written in the form $R \sin(\theta + \alpha)$ for $0 < \alpha < 90°$

[3 marks]

Question 5

(i) Expand the brackets; $\cos(\theta + \alpha)$

[1 mark]

(ii) Write $9 \cos\theta - 12 \sin\theta$ in the form $R \cos(\theta + \alpha)$ for $0 < \alpha < 90°$
Give α accurate to three significant figures.

[3 marks]

(iii) Solve the equation $9 \cos\theta - 12 \sin\theta = 15$ for $0 < \theta < 360°$

[3 marks]