5.1 The Remainder Theorem

Consider the following algebraic long division,

$$\begin{array}{r|l} & \quad\;\; x^2 - 4x - 32 \\ \hline x + 1 & x^3 - 3x^2 - 36x - 48 \\ & x^3 + x^2 \\ \hline & \quad - 4x^2 - 36x \\ & \quad - 4x^2 - 4x \\ \hline & \qquad\quad - 32x - 48 \\ & \qquad\quad - 32x - 32 \\ \hline & \qquad\qquad\quad - 16 \end{array} \quad \leftarrow \text{ Remainder}$$

It shows the cubic $p(x) = x^3 - 3x^2 - 36x - 48$ being divided by $(x + 1)$ and that the remainder is -16

The calculation is telling us is that,

$$\frac{x^3 - 3x^2 - 36x - 48}{x + 1} = x^2 - 4x - 32 + \frac{-16}{x + 1}$$

The remainder theorem gives a quick method of finding out what the remainder at the end of the algebraic long division will be without having to do the division.

It says,
"To find the remainder when $p(x)$ is divided by $(x + 1)$ calculate $p(-1)$"

$$\begin{aligned} p(-1) &= (-1)^3 - 3 \times (-1)^2 - 36 \times (-1) - 48 \\ &= -1 - 3 + 36 - 48 \\ &= -16 \end{aligned}$$

The Remainder Theorem

When a polynomial $p(x)$ is divided by $(x - a)$, where a is a constant,

the remainder is $p(a)$

Notice that, in the example, saying the remainder is – 16 meant $\dfrac{-16}{x + 1}$

5.2 Why Does The Remainder Theorem Work ?

As observed previously, from the long division,

$$\frac{x^3 - 3x^2 - 36x - 48}{x + 1} = x^2 - 4x - 32 + \frac{-16}{x + 1}$$

If this is multiplied throughout by (x + 1) it becomes,

$$x^3 - 3x^2 - 36x - 48 = (x^2 - 4x - 32)(x + 1) - 16$$

Now, putting the number 1 into the LHS is the same as putting it into the RHS.

On the RHS $(x + 1)$ then becomes zero and all that's left is the – 16

5.3 Example

$f(x) = x^3 - ax^2 + bx - 4$, where a and b are constants to be found

- The remainder when $f(x)$ is divided by $(x - 1)$ is 2
- The remainder when $f(x)$ is divided by $(x - 2)$ is also 2

Teaching Video : http://www.NumberWonder.co.uk/v9029/5.mp4

[5 marks]

5.4 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 40

Question 1

$f(x) = 4x^3 + ax^2 + bx - 27$, where *a* and *b* are constants to be found

- The remainder when $f(x)$ is divided by $(x - 1)$ is -2
- The remainder when $f(x)$ is divided by $(x + 1)$ is -100

[5 marks]

Question 2

Find the values of the constants *a* and *b* if the function

$$f(x) = a x^3 + b x^2 - 28x + 15$$

- is exactly divisible by $(x + 3)$
- leaves a remainder of – 60 when divided by $(x - 3)$

[5 marks]

Question 3

Additional Mathematics Examination Question from June 2014, Q6 (OCR)

The function $f(x) = x^3 - 4x^2 + ax + b$ is such that

- $x = 3$ is a root of the equation $f(x) = 0$
- when $f(x)$ is divided by $(x - 1)$ there is a remainder of 4

(i) Find the value of a and the value of b

[4 marks]

(ii) Solve the equation $f(x) = 0$

[3 marks]

Question 4

Additional Mathematics Examination question from June 2007, Q11 (OCR)

(a) You are given that

$$f(x) = x^3 - 3x^2 - 4x$$

(i) Find the three points where the curve $y = f(x)$ cuts the x-axis

[4 marks]

(ii) Sketch the graph of $y = f(x)$

[1 mark]

(b) You are given that

$$g(x) = x^3 - 3x^2 - 4x + 12$$

(i) Find the remainder when $g(x)$ is divided by $(x + 1)$

[2 marks]

(ii) Show that $(x - 2)$ is a factor of $g(x)$

[1 mark]

(iii) Hence solve the equation $g(x) = 0$

[4 marks]

Question 5

$$p(x) = x^4 + ax^3 + bx^2 + cx + d$$

The polynomial $p(x)$ has factors x and $(x + 1)$

(i) Prove that $a - b + c = 1$

[2 marks]

(ii) If the remainder when $p(x)$ is divided by $(x - 1)$ is 5, find the value of b

[3 marks]

Question 6

$$p(x) = 9x^2 + ax + b$$

$p(x)$ has remainder 12 when divided by either $(x - 1)$ or $(x + 2)$

(a) **(i)** Find the values of *a* and *b*

[2 marks]

(ii) Hence obtain the remainder when $p(x)$ is divided by $(x + 3)$

[2 marks]

(b) Prove that $9x^3 + ax^2 + bx - 6$ is divisible by $(x + 1)$

[2 marks]