Shrewsbury School
Mathematics Faculty

The Arnold Hagger Prize Paper 2004

Do as many questions as you can, in whatever order you like.

Do not (necessarily) expect to finish the paper!

Marks will be awarded for clarity and elegance of solutions, as well as correct answers.

Calculators are not allowed (and wouldn't be much help anyway, frankly).

1. If you were to write the following numbers out in full, what would the last digit be?

   (a) 2^{100} (b) 3^{333} (c) $1! + 2! + 3! + 4! + \ldots\ldots + 100!$

   [Note: n! means n x (n – 1) x (n – 2) x ….. x 3 x 2 x 1
   e.g. 3! = 6, 4! = 24 etc.]

   [5 marks]

2. Find all the prime numbers p for which $17p+1$ is a perfect square.

   [5 marks]

3. Show that the number x which satisfies the equation $2^x = 3$ cannot be rational.

   [5 marks]

4. (a) Prove that the five digit number ABCDE is divisible by 9 if $A+B+C+D+E$ is divisible by 9

   [5 marks]

5. (a) A statistician has two children, the elder of which is called Ronald. What is the probability that both children are boys?

   (b) A statistician has two children, one of which is called Ronald. What is the probability that both children are boys?

   [5 marks]

6. The function $C(n,r)$ is defined for non-negative integers n and r (with $n \geq r$) by:

   $$C(n,r) = \frac{n!}{(n-r)!r!}$$

   where the factorial function, n! , is defined as in Question 1.

   Show that $C(n,r\text{-}1) + C(n,r) = C(n+1,r)$.

   [5 marks]

7. The set {a, b, c} has 8 subsets: { } (the 'empty set'), {a}, {b}, {c}, {a, b}, {a, c}, {b, c}and finally {a, b, c}itself.

   (i) How many subsets does the set {a, b, c, d} have?

   (ii) How many subsets are possessed by a set containing n elements? (You should justify your answer convincingly).

   [5 marks]

8. Evaluate $\sqrt{6+\sqrt{6+\sqrt{6+\sqrt{6+.....}}}}$

[7 marks]

9. Here is a question from a Shrewsbury School Mathematics Prize paper from 1890:
In a boat race 1 mile in length, crew A beat crew B by 10 yards rowing downstream. Crew A beat crew C by 60 yards also rowing downstream. If it takes $1\frac{3}{4}$ times longer to row upstream than to row downstream, by how much would crew B beat crew C if the race were rowed upstream?

[8 marks]

10. Solve this puzzle, where each letter stands for a different number:

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   S E N D
 + M O R E
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 = M O N E Y
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[10 marks]

11. (a) Starting with the (obvious) inequality : $(a-b)^2 \geq 0$,
prove that $\sqrt{ab} \leq \dfrac{a+b}{2}$ for all real numbers a and b.
(b) Prove that $p^2+q^2+r^2 \geq pq+pr+qr$ for all real numbers p, q and r.

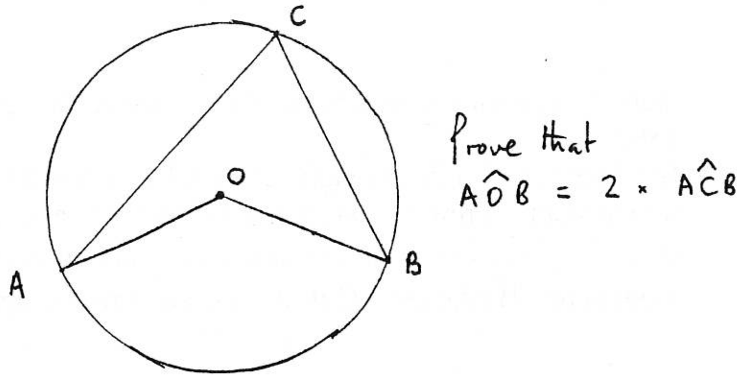
[10 marks]

12. In base 10, our usual number base, the number 123 means $1\times10^2+2\times10+3$.
In base 6, the number 123 means $1\times6^2+2\times6+3$ (equal to 51 base 10).

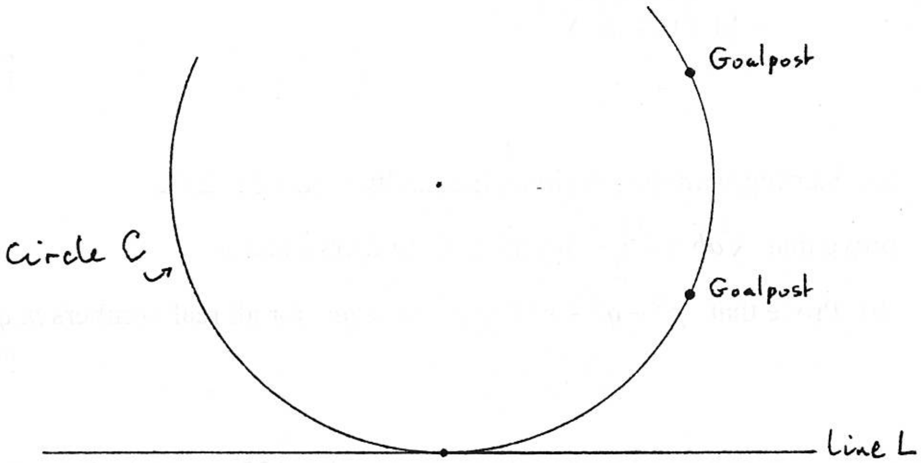
(a) Show that the number written as 10201 is always a perfect square in any base.
(b) Show that the number written as 10101 is always composite in any base.

[10 marks]

13. Prove that the angle subtended at the centre of a circle, O, by two points on the circumference of a circle, A and B, is twice that subtended by those points at another point, C, which is on the circumference and in the same sector as O:



(b) England rugby hero Johnny Vegas wants to place his rugby ball for a conversion. He must place it somewhere on the line L. Show that, in order to maximise the angle subtended by the goalposts, he should place it at the point where the line L just touches the circle C which passes through the goalposts and which has the line L as a tangent:



[10 marks]

14. A new method of measuring the distance between two points is devised. It is simply the sum of the difference in x co-ordinates and the difference in y co-ordinates.

e.g. the distance from (1, 3) to (5, 2) is 4 + 1 = 5 units
the distance from (2, -1) to (2, 5) is 0 + 6 = 6 units.

For each of the following, sketch the set of points on a diagram:

(i) Find the set of points a distance of 4 units from the origin.

(ii) Find the set of points equidistant from (0, 0) and (4, 4).

[10 marks]

MAO
Jan 2004