THE ARNOLD HAGGER MATHEMATICS PRIZE 1991

Wednesday 20th February 1991 7.45 to 9.15 p.m.

(Set by FMH)

Answer as many questions as you can, in any order. Do not expect to complete the paper: you are doing well if you can finish only a very few of the questions. The questions are not necessarily arranged in order of difficulty.

For each part of each question marks are given for Answer (A) and Method (M): the marks allocated are given in brackets after each part of each question, and the total number of marks per question is also given. You can see that it is important to explain your methods properly.

There are a total of 182 marks altogether, so you have lots to go at!

The results will be published in the form of a class-list, the names in each class below Class 1 being in alphabetical order to save embarrassment.

There will be lots of prizes in addition to ***the*** prize.

P.T.O.

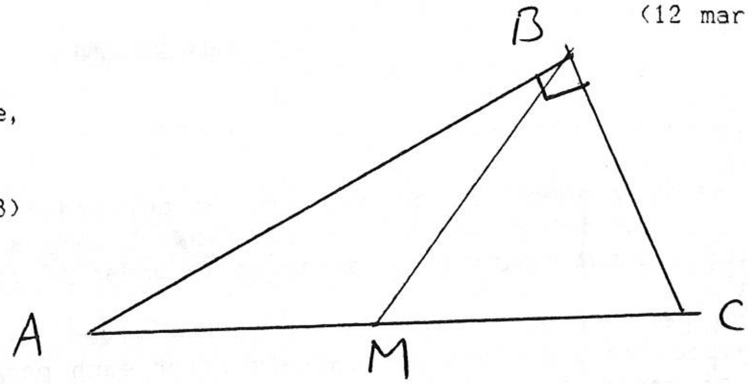
1. 27 equal unpainted wooden cubes are put together to make one large cube and then the 6 outside faces of this cube are painted red.

If I now dismantle the large cube and pick a small cube at random, and then pick a face of this cube at random, what is the probability that the chosen face is painted red? (A3, M3)

If I know that at least one face of the chosen small cube was painted red what now is the probability that the chosen face is painted red? (A2, M4)

(12 marks)

2. ABC is a right-angled triangle, and M is the mid-point of AC. Prove that MB = MA. (A0, M8)



(8 marks)

3. You will find that $f(n) = n^2 + n + 41$, where n is a positive integer, is a prime number for all small values of n. [f(1) = 43, f(2) = 47, f(3) = 53, f(4) = 61, f(5) = 71, and so on.]
Of course f(41) is not prime.
Find a value of n less than 41 for which f(n) is not prime, and prove that it is not prime, ***without using a calculator or working f(n) out.*** (A3, M5)
What special type of number is f(n) for this value of n, and why? (A2, M3)

(13 marks)

4. For what range of values of b (in terms of a) is $1/b < 1/a$

(i) if $a > 0$ (A3, M3)
(ii) if $a < 0$? (A3, M3)

For what range of values of b (in terms of a) is $1/b^2 < 1/a^2$? (A2, M2)

(16 marks)

5. Prove that there is no solution of $x^2 - 4y^2 = 1$ where x and y are non-zero integers. (A0, M8)

If one or both of x and y are allowed to be zero, what solutions are there? (A4, M0)

(12 marks)

6. By writing any positive integer as either 2k or 2k + 1, or otherwise, show that every perfect square of a positive integer is of one of the forms 4n or 4n + 1. (A0, M4)

Show also that every such square is of one of the forms 3m or 3m + 1. (A0, M6)

Why cannot such a perfect square be of the form 12p + 2 or 12p + 11? (A0, M6)

(16 marks)

7. Prove that $\dfrac{1}{k} - \dfrac{1}{k+1} = \dfrac{1}{k(k+1)}$ (A0, M4)

By working the opposite way and expressing $\dfrac{1}{1.2}, \dfrac{1}{2.3}, \dots \dfrac{1}{99.100}$ each as the difference of two fractions, or otherwise, show that

$$\frac{1}{1.2} + \frac{1}{2.3} + \dots + \frac{1}{99.100} = \frac{99}{100}$$

(A0, M6)

What is the "sum to infinity" of the series

$$\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \dots \ ?$$

(A2, M2)

(14 marks)

8. The Arithmetic Mean m of 2 positive numbers a and b is of course ½(a + b). Their Geometric Mean g is defined to be √(ab).
Prove that $m \geq g$. (A0, M6)
[You may find the inequality $(a - b)^2 \geq 0$ helpful!]

When does m = g? (A2, M1)

The ***Harmonic Mean*** h of a and b is defined by

$1/h = ½(1/a + 1/b)$.

Prove that $h \leq g$. (A0, M6)

(15 marks)

9. Show that $(ac + bd)^2 + (ad - bc)^2 = (a^2 + b^2)(c^2 + d^2)$. (A0, M4)

A ***Pythagorean Triple*** (x,y,z) is a set of three positive integers x,y and z such that x,y,z are the sides of a right angled triangle (we will assume that z is the hypotenuse).
Two such triples are (3,4,5) and (5,12,13). From the first part of the question I can deduce that (16,63,65) is another Pythagorean Triple. How are 16, 63 and 65 related to the numbers of the two original triples (3,4,5) and (5,12,13)? (A6, M4)

Find a Pythagorean Triple whose hypotenuse (the z value) is 845. (A4, M2)

(20 marks)

10. It is well known that the product of ***any*** three consecutiuve positive integers is divisible by 6. [You need not prove this. For example 3x4x5 + 60 = 6x10, 7x8x9 = 504 = 6x84.]

(i) What is the ***largest*** integer n which divides exactly into the product of ***any*** <u>six</u> consecutive positive integers? (A6, M0)

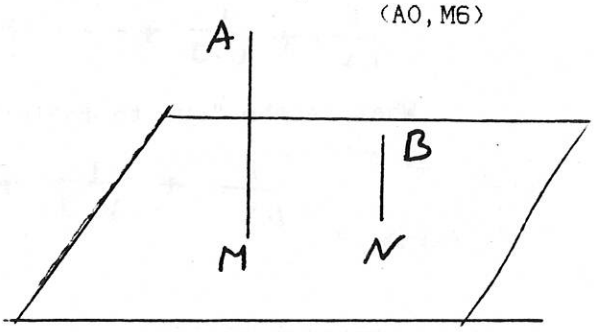
(ii) Prove that n does always divide into this product. (A0, M8)

(iii) Prove that there cannot be an integer greater than n with this property. (A0, M4)

(18 marks)

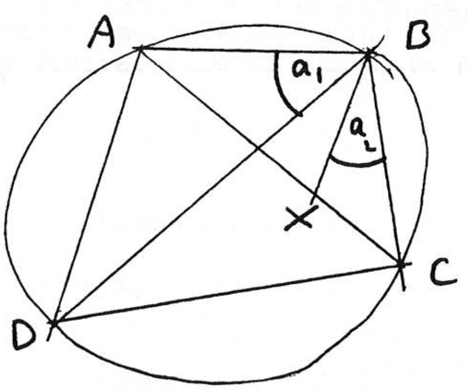
11. Let three planes π_1, π_2, π_3 meet in a single common point X. If π_2 and π_3 intersect in the line l_1, π_3 and π_1 intersect in the line l_2, and π_1 and π_2 intersect in the line l_3, explain why l_1, l_2 and l_3 all pass through X. (A0, M6)

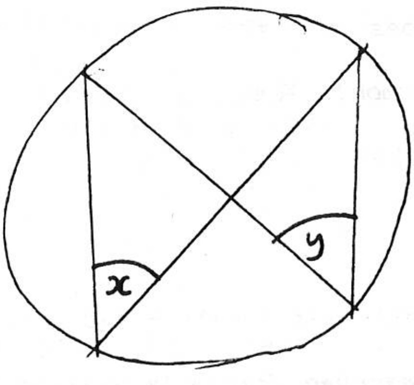
Two vertical nails AM and BN, of different lengths, protrude from a flat desk as shown. A large piece of card is placed across A and B and rests on the desk, meeting it in a line l. Show that, whatever the position of the card (so long as it is touching the nails at both A and B), l (extended if necessary) always passes through a certain fixed point, and identify the position of this point. (A2, M12)



(20 marks)

12.





ABCD is a quadrilateral inscribed in a circle and BX is drawn to make angle a_2 = angle a_1 as in the figure. Using the fact that angles in the same segment of a circle are equal (e.g. x = y in the second figure), show that triangles BAD and BXC are similar. (A0, M4)

Deduce that AD x BC = BD x CX. (A0, M4)

By considering triangles BDC and BAX show that AB x CD = BD x AX. (A0, M6)

Deduce Ptolemy's Theorem that for any cyclic quadrilateral

AD x BC + AB x CD = BD x AC. (A0, M4)

(18 marks)