

DBA  
DATE

SHREWSBURY SCHOOL MATHEMATICS PRIZE

1977 SDB.

YOU MUST EXPLAIN YOUR ANSWERS (No credit for answers without clear reasons)

1. Of two workmen, Andrew could complete a certain job in 12 days, whereas Basil would take 15 days. Andrew works alone at the job for three days and then is joined by Basil. How long will they have to work together to complete the job, assuming that each works at his normal rate.
2. Three equal circles pass through a point P. Their other points of intersection are A, B and C. Show that the circle through A, B and C is equal to the others.
3. Any four consecutive, positive whole numbers are taken. The differences (taken positively) between the product of one pair and the product of the other pair are found in all possible ways. Show that the largest of these differences is always the product of the other differences.
4. Find all the ways of arranging four points in a plane so that there are at most two different values for the distance between any pair of them
5. (i) If  $a + b + c + d = 0$  prove that  
 $(a + b)^3 + (a + c)^3 + (a + d)^3 + (b + c)^3 + (b + d)^3 + (c + d)^3 = 0$   
(ii) If  $x^2 = x + 5$  prove that  $x^5 = 41x + 55$ .
6. Find the missing digits in the following multiplication

$$\begin{array}{r} 4 * * \\ 3 * \\ \hline 36 * * \\ * * 7 * \\ \hline * * 3 * * \end{array}$$

Each of questions 7 to 12 poses a situation about which several statements are made. Some of these statements are true and some are false. You are required to say of each statement whether it is true or false and to give brief reasons for your answer.

7. 130 postcards are distributed among seven pigeonholes.
- (i) There must be some pigeonhole which contains at least 19 postcards.
  - (ii) There must be some pigeonhole which does not contain at least 19 postcards.
  - (iii) There cannot be a pigeonhole which contains a number of postcards equal to the **average** number of postcards in the other six pigeonholes.
8. A palindromic number is one which reads the same backwards as forwards, such as 12321.
- (i) All palindromic numbers with an even number of digits are divisible by 11.
  - (ii) All powers of 11 are palindromic.
  - (iii) There are no four digit numbers which are palindromic and perfect squares.
9. Nine square tiles, each of area one square foot, are placed together to form a square with a 3 foot side, An ant crawls along the edges of the tiles and in the course of his journey he visits (at least once) every point on the edge of every tile. If the length  $l$  of his journey is as small as possible then
- (i) his journey must end at his start point.
  - (ii)  $l = 27$  feet
  - (iii)  $l = 28$  feet.

10. If the three coplanar triangles OAB, COB and CDO are directly similar (corresponding vertices being determined by the order) then it follows that
- A, B, C and D lie on a circle.
  - A, O and D are collinear. (Lie on a line)
  - OA = OD.
11. A cube is to be coloured with  $n$  (less than or equal to 6) colours so that no two faces with an edge in common receive the same colour. Two cubes are said to have the same colouring if, by rotating one of them if necessary, they can be made to look identical.
- If  $n = 6$ , there are 30 distinct colourings which use all the colours.
  - $n$  must be at least 4.
  - If  $n = 4$ , the number of distinct colourings is 6.
12. A farmer uses  $n$  equal lengths of fencing to form a paddock. The fences do not intersect and are straight and are fixed to  $n$  posts. For technical reasons the angles between the fences can only take the values  $45^\circ$ ,  $60^\circ$ ,  $90^\circ$  and  $120^\circ$ .
- If  $n = 3$  all the interior angles are  $60^\circ$
  - If  $n = 4$ , one interior angle could be  $45^\circ$
  - $n$  cannot equal 5.
  - If exactly  $m$  corners point inwards,

$$\frac{24 + 13m}{9} \leq n \leq \frac{24 + 13m}{4}$$